MOTIVIC SYMMETRIC RING SPECTRUM REPRESENTING ALGEBRAIC K-THEORY

BY

YOUNGSOO KIM

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2010

Urbana, Illinois

Doctoral Committee:

Professor Randy McCarthy, Chair
Professor Emeritus Daniel R. Grayson, Director of Research
Associate Professor Mathew Ando
Associate Professor Charles Rezk

Abstract

Voevodsky showed that there is a motivic spectrum representing algebraic K-theory. We describe an equivalent spectrum that is also a symmetric ring spectrum. A coherence problem occurs when one verifies the symmetry. It is explained and solved by introducing a category of vector bundles with strictly associative tensor product that is also strictly commutative with line bundles.

To Father and Mother.

Acknowledgment

I would like to express my gratitude to all those who gave me the possibility to complete this dissertation. I want to thank the Department of Mathematics at the University of Illinois at Urbana-Champaign for providing me financial support and resources during my graduate study. I am deeply indebted to my advisor Dan Grayson for sharing his knowledge, ideas, and for being patient while I was not making much progress in my research. I also would like to thank my thesis committee members, Randy McCarthy, Matt Ando, and Charles Rezk. Oliver Röndigs refereed the draft of this dissertation and pointed out numerous errors I made. I especially thank him for valuable comments.

I also want to thank the University of Bielefeld and the SonderForschungsBereich 504 under the direction of Ulf Rehmann and Marcus Rost, where my advisor learned about this problem from Ivan Panin, whom I also thank.

Finally, I thank my family for their unconditional support and love, especially my wife, Jieun Park.

Table of Contents

Chapter 1

Introduction

The motivic homotopy theory introduced in [17] and [24] is the homotopy theory of schemes. In [24], Voevodsky gave three examples of cohomology theories represented by motivic spectra. One of them is algebraic K-theory, which is represented by a spectrum he called BGL. Then Panin, Pimenov, and Röndigs proved that this spectrum has a monoidal structure which respects the naive multiplicative structure of K-theory [20].

Voevodsky himself said in [24] that the construction of BGL is ugly, and that he is sure there will be a better way to do it. One of the reasons is that the structure maps are defined only up to homotopy. In this paper we introduce a new construction of a motivic spectrum representing algebraic K-theory, which is equivalent to BGL, but whose structure maps are defined explicitly. As a result, it is rather straightforward to prove that the spectrum has a monoidal structure. In particular, it is a motivic symmetric ring spectrum (see Definition 2.3.12). The same result was announced in a recent preprint by Röndigs, Spitzweck, and Østvær [18]. There are several differences between their spectrum and ours. Their spectrum is advantageous in that it is commutative, whereas ours is not. Also they proved theirs is equivalent to Voevodsky's spectrum for non-regular base schemes, but we prove it only for regular base schemes. On the other hand, our construction is purely K-theoretic using fewer outside result, and the construction and proofs are elementary. It remains as an interesting future project to prove the equivalence of our spectrum with Voevodsky's spectrum for non-noetherian bases, or to construct a commutative symmtric ring spectrum using similar techniques.

In chapter 2, we review the theory of motivic symmetric spectra of Jardine [13], which is the main technical basis of the construction in this paper.

In chapter 3, the category of *standard vector bundles* is constructed. It is a category which is equivalent to the usual category of vector bundles of a scheme, but has nice properties listed in Theorem 3.3.10 and Theorem 3.3.8. In particular, the tensor product of vector bundles is strictly associative and strictly commutative with line bundles. The strict associativity is required to construct the monoidal structure of the spectrum, and the strict commutativity with line bundles is required to prove the equivariance of the structure maps.

In chapter 4, we review the Gillet-Grayson construction of K-theory (the G-construction for short), which is the main ingredient of the construction of the motivic symmetric ring spectrum representing algebraic K-theory. What makes an explicit definition of the structure map possible is the property of the G-construction that the multiplication by arbitrary elements of K_0 such as $[\mathcal{O}] - [\mathcal{O}(-1)] \in K_0(\mathbb{P}^1)$ can be described explicitly, which cannot be done with the Quillen Q-construction or Waldhausen K-theory, although Quillen's $S^{-1}S$-construction would work as well.

Finally in chapter 5, we construct the spectrum and show that it is a motivic symmetric ring spectrum. The definition is simple. The n-th space of the spectrum is the presheaf $X \mapsto \mathrm{diag} G^n \mathbf{V}(X)$ assigning the iterated G-construction of the category of standard vector bundles on X to each scheme X, and the structure maps are defined by multiplication by the element $[\mathcal{O}] - [\mathcal{O}(-1)]$ in K_0. Also, it is shown that the spectrum is equivalent to BGL when the base scheme is regular.

For the purpose of this paper, we will restrict our attention to certain small categories of schemes instead of the large category $((Sch))$ of all schemes. We choose an uncountable cardinal κ that is big enough for any application of motivic homotopy theory. In particular, κ has to be bigger than the cardinality of the continuum for the theory over the complex numbers. Consider the category of noetherian schemes of finite Krull dimension whose coordinate rings have cardinality less than κ. It is equivalent to a small category, and we let Sch denote this small category. We may assume that Sch has all open subschemes of all of its objects. When we mention a scheme in this paper, we mean an object of Sch. Suppose X is a scheme. We let Sch/X denote the category of schemes over X and let Sm/X denote the category of smooth schemes (of finite type) over X. They are small categories, and Sm/X is a subcategory of Sch/X. Any scheme in the large category $((Sch))$ that has a smooth map over a scheme (in Sch) is also noetherian, finite dimensional, and the cardinalities of its coordinate rings are less than κ. Hence Sm/X is equivalent to the large category of smooth schemes in $((Sch))$ over X.

Chapter 2

Motivic symmetric spectra

Hovey, Shipley, and Smith introduced a symmetric monoidal category of spectra called the category of symmetric spectra [11], and Jardine imported the theory to the motivic stable category of Morel and Voevodsky [13, 17, 24]. The theory provides a model of the motivic stable category with a symmetric monoidal smash product.

In this chapter, we summarize his theory of motivic symmetric spectra. The content of this chapter is not original, and it is drawn from [13, 17, 24, 11, 12, 22]. The category of motivic symmetric spectra is defined, and the result that it can be considered as a symmetric monoidal model category is presented. The associated homotopy category is equivalent to the motivic stable homotopy category of Morel and Voevodsky.

2.1 Motivic symmetric spectra

Definition 2.1.1. A *spectrum* is

(1) a sequence $X_0, X_1, \dots, X_n, \dots$ of pointed simplicial sets,

(2) a pointed map $\sigma : S^1 \wedge X_n \to X_{1+n}$ for each $n \geq 0$.

The maps σ are called the *structure maps* of the spectrum. By adding symmetric group actions to a spectrum, we get the notion of symmetric spectrum. Let Σ_p be the symmetric group of permutations of the set $\{1, 2, \dots, p\}$. We consider $\Sigma_p \times \Sigma_n$ as a subgroup of Σ_{p+n} with Σ_p acting on the first p elements and Σ_n on the last n elements. Let $S^p = S^1 \wedge \cdots \wedge S^1$ be the p-fold smash power of the simplicial circle S^1. Then Σ_p acts on S^p by permuting the coordinates.

Definition 2.1.2. [11, 1.2.2] A *symmetric spectrum* is

(1) a sequence $X_0, X_1, \dots, X_n, \dots$ of pointed simplicial sets,

(2) a pointed map $\sigma : S^1 \wedge X_n \to X_{1+n}$ for each $n \geq 0$, and

(3) a (base point preserving) left action of the symmetric group Σ_n on X_n for each $n \geq 0$ such that the composite map

$$\sigma^p : S^p \wedge X_n \to S^{p-1} \wedge X_{1+n} \to \cdots \to S^1 \wedge X_{p-1+n} \to X_{p+n}$$

of the maps $Id_{S^i} \wedge \sigma : S^i \wedge S^1 \wedge X_{p-i-1+n} \to S^i \wedge X_{p-i+n}$ is $\Sigma_p \times \Sigma_n$-equivariant for $p \geq 0$ and $n \geq 0$.

These notions can be defined in the motivic world as well by replacing simplicial sets by *motivic spaces*. We follow Panin, Pimenov, and Röndigs for the definition of motivic spaces [20, 1.1]. Let $(Sm/S)_{Nis}$ be the site of smooth schemes over a base scheme S with Nisnevich topology, and let **SSet** be the category of simplicial sets.

Definition 2.1.3 ([20]). A *motivic space over* S is a functor $(Sm/S)^{op} \to \mathbf{SSet}$, i.e., a presheaf of simplicial sets (or simplicial presheaf) on Sm/S. The category of motivic spaces over S is denoted by $\mathbf{M}(S)$.

Morel and Voevodsky used simplicial sheaves on $(Sm/S)_{Nis}$ instead of simplicial presheaves for motivic homotopy theory. Theorem 1.2 of [13] shows that the presheaf category is also a good model for motivic homotopy theory, provided one adjusts the choice of weak equivalences appropriately.

We have two faithful functors $Sm/S \to \mathbf{M}(S)$ and $\mathbf{SSet} \to \mathbf{M}(S)$. The first functor sends a smooth scheme X to the presheaf of discrete simplicial sets $Y \mapsto \mathrm{Hom}_{Sm/S}(Y, X)$ and the second functor sends a simplicial set K to the constant presheaf $Y \mapsto K$. Smooth schemes and simplicial sets will be considered as motivic spaces by means of these functors.

A *pointed motivic space* is a motivic space A with a map $a_0 : S \to A$ called the base point of A. We usually omit the base point and write A instead of (A, a_0). The category of pointed motivic spaces will be denoted by $\mathbf{M}_\bullet(S)$. There is the forgetful functor $\mathbf{M}_\bullet(S) \to \mathbf{M}(S)$ forgetting the base point. Its adjoint $\mathbf{M}(S) \to \mathbf{M}_\bullet(S)$ is defined by $A \mapsto A_+$ where $A_+ = A \coprod S$ with the canonical map $S \to A \coprod S$ for the base point. The smash product $A \wedge B$ of pointed motivic spaces A and B is defined to be the presheaf taking Y to the smash product of pointed simplicial sets $A(Y)$ and $B(Y)$.

Lemma 2.1.4. *Let $X \in Sm/S$ and let $ev_X : \mathbf{M}_\bullet(S) \to \mathbf{SSet}_\bullet$ be the evaluation functor $A \mapsto A(X)$. There is an adjunction*

$$(w_X, ev_X) : \mathrm{Hom}_{\mathbf{M}_\bullet(S)}(K \wedge X_+, A) \cong \mathrm{Hom}_{\mathbf{SSet}_\bullet}(K, A(X)).$$

Proof. If $\alpha : K \wedge X_+ \to A$ is a map of pointed motivic spaces, we obtain a map $K \to A(X)$ defined by $x \mapsto \alpha(x, 1_X)$. Conversely, given a map $h : K \to A(X)$ of simplicial sets, we obtain $K \wedge X_+ \to A$ by defining, for each $Y \in Sm/S$, $(K \wedge X_+)(Y) \to A(Y)$ to be the map sending (x, f) to $A(f)hx$ for

$f : Y \to X$, and to the base point for $f : Y \to S$. $\square$

The category $\mathbf{M}_\bullet(S)$ of motivic spaces over S is a closed symmetric monoidal category with respect to the smash product, meaning that there is an internal function object. Suppose A, B, and C are pointed motivic spaces. The *function complex* $\mathbf{hom}(A, B)$ is a simplicial set defined by $n \mapsto \mathrm{Hom}_{\mathbf{M}(S)}(A \times \Delta^n, B)$. The pointed version is $\mathbf{hom}_\bullet(A, B)$ defined by $n \mapsto \mathrm{Hom}_{\mathbf{M}_\bullet(S)}(A \wedge (\Delta^n)_+, B)$. The *internal hom complex* $\mathbf{Hom}(A, B)$ is a motivic space defined by $U \mapsto \mathbf{hom}(A|_U, B|_U)$. The pointed version is $\mathbf{Hom}_\bullet(A, B)$ defined by $U \mapsto \mathbf{hom}_\bullet(A|_U, B|_U)$. The base point of $\mathbf{Hom}_\bullet(A, B)$ is determined by the maps $A \wedge (\Delta^n)_+ \to S \xrightarrow{b_0} B$. There is an adjoint isomorphism

$$\mathrm{Hom}_{\mathbf{M}_\bullet(S)}(A \wedge B, C) \cong \mathrm{Hom}_{\mathbf{M}_\bullet(S)}(A, \mathbf{Hom}_\bullet(B, C)).$$

Evaluation in U-sections defines natural maps $\mathbf{hom}_\bullet(A|_U, B|_U) \wedge A(U) \to B(U)$, which together gives a natural *evaluation map*

$$ev : \mathbf{Hom}_\bullet(A, B) \wedge A \to B.$$

We denote by T^p the p-fold smash product $T \wedge T \wedge \cdots \wedge T$ of a pointed motivic space T. In the following definition, T could be any pointed motivic space, but in motivic homotopy theory, it is usually a space from the class of *motivically equivalent* spaces that includes $\mathbb{A}^1_S/\mathbb{A}^1_S - \{0\}$, $(\mathbb{P}^1_S, \infty)$, and $S^1 \wedge \mathbb{G}_m$. Our choice for T in Chapter 5 will be the mapping cylinder of the inclusion $S \xrightarrow{\infty} \mathbb{P}^1_S$ of the point at infinity. More will be explained about the choice in Chapter 5.

Definition 2.1.5. A *motivic T-spectrum over S* is

(1) a sequence $A_0, A_1, \ldots, A_n, \ldots$ of pointed motivic spaces over S,

(2) a pointed map $\sigma : T \wedge A_n \to A_{1+n}$ for each $n \geq 0$.

Definition 2.1.6 ([13])**.** A *motivic symmetric T-spectrum over S* is

(1) a sequence $A_0, A_1, \ldots, A_n, \ldots$ of pointed motivic spaces over S,

(2) a pointed map $\sigma : T \wedge A_n \to A_{1+n}$ for each $n \geq 0$, and

(3) a (base point preserving) left action of the symmetric group Σ_n on A_n for $n \geq 0$, such that the composite map

$$\sigma^p : T^p \wedge A_n \to T^{p-1} \wedge A_{1+n} \to \cdots \to T \wedge A_{p-1+n} \to A_{p+n}$$

of the maps $Id_{T^i} \wedge \sigma : T^i \wedge T^1 \wedge A_{p-i-1+n} \to T^i \wedge A_{p-i+n}$ is $\Sigma_p \times \Sigma_n$-equivariant for $p \geq 0$ and $n \geq 0$.

A morphism f between motivic T-spectra A and B is a collection of maps $f_n : A_n \to B_n$ such that the diagram

$$\begin{tikzcd} T \wedge A_n \arrow[r, "\sigma"] \arrow[d, "1\wedge f_n"'] & A_{1+n} \arrow[d, "f_{1+n}"] \\ T \wedge B_n \arrow[r, "\sigma"'] & B_{1+n} \end{tikzcd}$$

commutes for each $n \geq 0$. The category of motivic T-spectra will be denoted by $\mathbf{SM}_T(S)$, or simply $\mathbf{SM}(S)$ when T is evident from the context. A morphism f between motivic symmetric T-spectra A and B is a collections of Σ_n-equivariant maps $f_n : A_n \to B_n$ such that a similar diagram commutes. The category of motivic symmetric T-spectra will be denoted by $\mathbf{SM}_T^\Sigma(S)$ or $\mathbf{SM}^\Sigma(S)$.

Lemma 2.1.7. *Suppose A is a motivic T-spectrum together with a symmetric group action on each A_n. If $\sigma : T \wedge A_n \to A_{1+n}$ is $\Sigma_1 \times \Sigma_n$-equivariant and $\sigma^2 : T^2 \wedge A_n \to A_{2+n}$ is $\Sigma_2 \times \Sigma_n$-equivariant for all $n \geq 0$, then A is a motivic symmetric T-spectrum. In other words, the equivariance of iterated structure maps σ^p needs to be proved only for $p \leq 2$.*

Proof. Let $(\alpha, \beta) \in \Sigma_p \times \Sigma_n$. Since the symmetric group Σ_p is generated by adjacent transpositions, it suffices to prove that $\sigma^p(\alpha, \beta) = (\alpha, \beta)\sigma^p$ assuming α is the transposition $(i \;\; i+1)$ for some $1 \leq i \leq p-1$. This is shown by the following commutative diagram where τ is the transposition such that
$\sigma = (1, \tau, 1) \in \Sigma_{i-1} \times \Sigma_2 \times \Sigma_{p-i-1}$.

$$\begin{tikzcd} T^{i-1} \wedge T^2 \wedge T^{p-i-1} \wedge A_n \arrow[r, "1\wedge\tau\wedge 1\wedge\beta"] \arrow[d, "1\wedge 1\wedge\sigma^{p-i-1}"] \arrow[ddd, bend right=60, "\sigma^p"'] & T^{i-1} \wedge T^2 \wedge T^{p-i-1} \wedge A_n \arrow[d, "1\wedge 1\wedge\sigma^{p-i-1}"'] \arrow[ddd, bend left=60, "\sigma^p"] \\ T^{i-1} \wedge T^2 \wedge A_{(p-i-1)+n} \arrow[r, "1\wedge\tau\wedge(1,\beta)"] \arrow[d, "1\wedge\sigma^2\wedge 1"] & T^{i-1} \wedge T^2 \wedge A_{(p-i-1)+n} \arrow[d, "1\wedge\sigma^2\wedge 1"'] \\ T^{i-1} \wedge A_{2+(p-i-1)+n} \arrow[r, "1\wedge(\tau,1,\beta)"] \arrow[d, "\sigma^{i-1}"] & T^{i-1} \wedge A_{2+(p-i-1)+n} \arrow[d, "\sigma^{i-1}"'] \\ A_{p+n} \arrow[r, "(\alpha,\beta)"'] & A_{p+n} \end{tikzcd}$$

The top and bottom squares commute because σ is equivariant and the middle square commutes because σ^2 is equivariant. $\square$

2.2 Model structures

The category of motivic symmetric spectra $\mathbf{SM}^{\Sigma}(S)$ has a stable model category structure. The associated homotopy category $H(\mathbf{SM}^{\Sigma}(S))$ is equivalent to the motivic stable homotopy category of Morel and Voevodsky.

We begin with the description of two model structures on $\mathbf{M}(S)$. One of them is called simplicial and the other is called motivic. A *simplicial weak equivalence* is a map of motivic spaces inducing weak equivalences of simplicial sets on all (Nisnevich) stalks. (Jardine calls such a map a *local* weak equivalence.) A *cofibration* is a monomorphism of motivic spaces, and a *simplicial fibration* is a map that has the right lifting property with respect to all maps that are cofibrations and simplicial weak equivalences. (Jardine calls such a map a *global* fibration.) These classes of maps define a model structure on $\mathbf{M}(S)$ [12, 2.3]. We call it the *simplicial* model structure and denote the associated homotopy category by $H^s(\mathbf{M}(S))$. The motivic model structure is obtained by localizing $\mathbf{M}(S)$ with respect to the map $f : * \to \mathbb{A}^1$ of a rational point. A map is called a *motivic fibration* if it is a simplicial fibration and has the right lifting property with respect to all inclusions of motivic spaces

$$(f, j) : (\mathbb{A}^1 \times A) \cup_A B \to \mathbb{A}^1 \times B$$

arising from $f : * \to \mathbb{A}^1$ and all cofibrations $j : A \to B$. A *cofibration* is a monomorphism of motivic spaces as in simplicial structure. A map $g : A \to B$ is called a *motivic weak equivalence* if it induces a weak equivalence of simplicial sets

$$g^* : \mathbf{hom}(B, C) \to \mathbf{hom}(A, C)$$

for every motivically fibrant object C. The category $\mathbf{M}(S)$ of motivic spaces over S is a model category, together with the classes of cofibrations, motivic weak equivalences and motivic fibrations [13, 1.1]. The simplicial and motivic model structures on the category $\mathbf{M}_\bullet(S)$ of pointed motivic spaces are induced from those of $\mathbf{M}(S)$. The motivic homotopy category will be denoted by $\mathbf{H}_\bullet(S)$.

Proposition 2.2.1. *A motivic space A is motivically fibrant if and only if it is simplicially fibrant and the projection $X \times_S \mathbb{A}^1_S \to X$ induces a weak equivalence of simplicial sets $A(X) \simeq A(X \times_S \mathbb{A}^1)$ for all X in Sm/S.*

Proof. We refer the reader to the discussion preceding [13, 1.6]. □

The homotopy (presheaf of) groups of a pointed motivic space A are defined as follows:

- $\pi_n^s(A) : X \mapsto \mathrm{Hom}_{H^s(\mathbf{M}_\bullet(S))}(S^n \wedge X_+, A)$,

- $\pi_n^{\mathbb{A}^1}(A) : X \mapsto \mathrm{Hom}_{\mathbf{H}_\bullet(S)}(S^n \wedge X_+, A)$.

Lemma 2.2.2. *Suppose A is a pointed motivic space. For all $n \geq 0$ and X in Sm/S, there are isomorphisms*

$$\pi_n^s(A)(X) \cong \pi_n((Ex^s A)(X)),$$

$$\pi_n^{\mathbb{A}^1}(A)(X) \cong \pi_n((Ex^{\mathbb{A}^1} A)(X)),$$

where Ex^s and $Ex^{\mathbb{A}^1}$ are simplicially fibrant replacement and motivically fibrant replacement functors, respectively.

Proof. The adjunction of Lemma 2.1.4 is a Quillen adjunction with respect to both model structures since the functor $K \mapsto K \wedge X_+$ preserves cofibrations and trivial cofibrations. Therefore, there are isomorphisms

$$\mathrm{Hom}_{H^s(\mathbf{M}_\bullet(S))}(K \wedge X_+, A) \cong \mathrm{Hom}_{H(\mathbf{SSet}_\bullet)}(K, (Ex^s A)(X))),$$
$$\mathrm{Hom}_{\mathbf{H}_\bullet(S)}(K \wedge X_+, A) \cong \mathrm{Hom}_{H(\mathbf{SSet}_\bullet)}(K, (Ex^{\mathbb{A}^1} A)(X)).$$

The lemma follows when $K = S^n$. $\square$

Now consider the category $\mathbf{SM}(S)$ of motivic T-spectra. There are two model structures on $\mathbf{SM}(S)$, the one with *level equivalences* as weak equivalences, and the one with *stable equivalences* as weak equivalences. We say that a map $f : A \to B$ of motivic T-spectra is a *level cofibration*, *level fibration*, and *level equivalence* if all component maps $f_n : A_n \to B_n$ are cofibrations, motivic fibrations, and motivic weak equivalences, respectively. A map in $\mathbf{SM}(S)$ is called a *cofibration* if it has the left lifting property with respect to all maps that are level fibrations and level equivalences. The category $\mathbf{SM}(S)$ of motivic T-spectra is a model category, together with the classes of level equivalences, cofibrations, and level fibrations [13, 2.1]. We write JA for the level fibrant model of a motivic spectrum A.

For a pointed motivic space A, let $\Omega_T A$ denote the internal hom functor $\mathbf{Hom}_\bullet(T, A)$. The *fake T-loop functor* $\Omega_T^\ell : \mathbf{SM}(S) \to \mathbf{SM}(S)$ is defined by setting, for a motivic T-spectrum A,

$$(\Omega_T^\ell A)_n = \Omega_T(A_n),$$

and by specifying the structure map $\sigma : T \wedge (\Omega_T^\ell A)_n \to (\Omega_T^\ell A)_{1+n}$ to be the map adjoint to $\Omega_T(\sigma_*) : \Omega_T(A_n) \to \Omega_T^2(A_{1+n})$ where $\sigma_* : A_n \to \Omega_T(A_{1+n})$ is adjoint to the composite

$A_n \wedge T \xrightarrow{t} T \wedge A_n \xrightarrow{\sigma} A_{1+n}$. The maps σ_* determine a natural morphism of motivic T-spectra

$$\sigma_* : A \to \Omega_T^\ell A[1],$$

where the shifted T-spectrum $A[1]$ is defined by $A[1]_n = A_{1+n}$. The motivic spectrum $Q_T A$ is defined to be the colimit of the system

$$A \xrightarrow{\sigma_*} \Omega_T^\ell A[1] \xrightarrow{\Omega_T^\ell \sigma_*[1]} (\Omega_T^\ell)^2 A[2] \xrightarrow{(\Omega_T^\ell)^2 \sigma_*[2]} \cdots$$

The functor Q_T is called the *stabilization functor*.

A map $f : A \to B$ of motivic T-spectra is said to be a *stable equivalence* if it induces a level equivalence

$$Q_T J(f) : Q_T JA \to Q_T JB.$$

A *stable fibration* is a map that satisfies the right lifting property with respect to all maps that are cofibrations and stable equivalences. The category $\mathbf{SM}(S)$ of motivic T-spectra over S is a model category, together with the classes of cofibrations, stable fibrations, and stable equivalences [13, 2.9]. This model structure will be called *the stable model structure*, and $\mathbf{SM}(S)$ is assumed to be equipped with this model structure unless stated otherwise. The associated homotopy category $H(\mathbf{SM}(S))$ is the *motivic stable homotopy category* of Morel and Voevodsky denoted by $\mathcal{SH}(S)$ in [17]. We will denote it by $\mathbf{SH}(S)$.

Finally, we will define *the stable model structure* on the category $\mathbf{SM}^\Sigma(S)$ of motivic symmetric T-spectra. A map in $\mathbf{SM}^\Sigma(S)$ is said to be a *level cofibration*, *level fibration*, and *level equivalence* if all component maps are cofibrations, motivic fibrations, and motivic weak equivalences, respectively. An *injective fibration* is a map that has the right lifting property with respect to all maps which are both level cofibrations and level equivalences. The category $\mathbf{SM}^\Sigma(S)$ is a model category, together with the classes of level cofibrations, level equivalences, and injective fibrations [13, 4.2]. A map in $\mathbf{SM}^\Sigma(S)$ is said to be a *stable fibration* if it is a stable fibration as a map of motivic T-spectra (forgetting symmetric group action). A map $f : A \to B$ is said to be a *stable equivalence* if it induces weak equivalence of Kan complexes

$$f^* : \mathbf{hom}_\bullet(B, C) \to \mathbf{hom}_\bullet(A, C)$$

for all injective stably fibrant objects C, and a map is called a *stable cofibration* if it satisfies the left lifting property with respect to all stable fibrations that are also stable equivalences. Then the category $\mathbf{SM}^\Sigma(S)$

of symmetric T-spectra over S is a model category, together with the classes of stable equivalences, stable fibrations, and stable cofibrations [13, 4.15]. This model structure is called *the stable model structure* on $\mathbf{SM}^{\Sigma}(S)$, and the category $\mathbf{SM}^{\Sigma}(S)$ is assumed to be equipped with this model structure unless stated otherwise. The associated homotopy category is denoted by $\mathbf{SH}^{\Sigma}(S)$. Let $U : \mathbf{SM}^{\Sigma}(S) \to \mathbf{SM}(S)$ be the forgetful functor. There is an adjoint called the symmetrization functor $V : \mathbf{SM}(S) \to \mathbf{SM}^{\Sigma}(S)$.

Theorem 2.2.3 ([13] 4.31)**.** *The functors U and V induce an adjoint equivalence of stable homotopy categories*

$$\mathbf{SH}^{\Sigma}(S) \rightleftarrows \mathbf{SH}(S).$$

Proposition 2.2.4. *Let A and B be motivic T-spectra in $\mathbf{SM}(S)$ with structure maps $\alpha_n : T \wedge A_n \to A_{n+1}$ and $\beta_n : T \wedge B_n \to B_{n+1}$. Suppose that B_n are motivically fibrant for all $n \geq 0$ and that there are isomorphisms $w_n : A_n \to B_n$ in the motivic homotopy category $\mathbf{H}_{\bullet}(S)$ such that the following diagram commutes.*

$$\begin{tikzcd} T \wedge A_n \arrow[r, "\overline{\alpha_n}"] \arrow[d, "1\wedge w_n"'] & A_{n+1} \arrow[d, "w_{n+1}"] \\ T \wedge B_n \arrow[r, "\overline{\beta_n}"'] & B_{n+1} \end{tikzcd}$$

Then there exists a motivic T-spectrum C and level equivalences $A \xleftarrow{\simeq} C \xrightarrow{\simeq} B$ in $\mathbf{SM}(S)$. In particular, A and B are equivalent spectra.

Proof. We construct C inductively. Define $C_0 = A_0$, and define C_1 to be the middle term of the factorization of the structure map $\alpha_0 : T \wedge A_0 \to A_1$ as $\varphi_1\gamma_0$

$$T \wedge A_0 = T \wedge C_0 \xrightarrow{\gamma_0} C_1 \xrightarrow{\varphi_1} A_1$$

with γ_0 a cofibration and φ_1 a trivial fibration. If $C_0, \dots, C_n$ has been defined with structure maps $\gamma_i : T \wedge C_i \to C_{i+1}$ for $0 \leq i \leq n-1$, define C_{n+1} to be the middle term of the factorization of the composite $\alpha_n(1 \wedge \varphi_n)$ as $\varphi_{n+1}\gamma_n$

$$\begin{tikzcd} & T \wedge A_n \arrow[dr, "\alpha_n"] & \\ T \wedge C_n \arrow[ur, "1\wedge\varphi_n"] \arrow[dr, "\gamma_n"'] & & A_{n+1} \\ & C_{n+1} \arrow[ur, "\varphi_{n+1}"'] & \end{tikzcd}$$

with γ_n a cofibration and φ_{n+1} a trivial fibration. The construction of C and the above diagram shows that $\varphi : C \to A$ is a level equivalence, and each structure map of C is cofibrant.

Next, we construct motivic weak equivalences $\psi_n : C_n \to B_n$ inductively. We remark that any map in the homotopy category with target B_n can be lifted to a map of motivic spaces since every motivic space is cofibrant and B_n is motivically fibrant (Proposition 5.11 [4]). We get the first map $\psi_0 : C_0 \to B_0$ by lifting the isomorphism $w_0 : C_0 = A_0 \to B_0$. It is a motivic weak equivalence being a lift of an isomorphism (Proposition 5.8 [4]). Consider the following commutative diagram.

$$\begin{tikzcd}
T \wedge C_0 \arrow[r, "\overline{\gamma_0}"] \arrow[d, equal, "1\wedge\overline{\varphi_0}"'] \arrow[dd, bend right=60, "1\wedge\overline{\psi_0}"'] & C_1 \arrow[d, "\overline{\varphi_1}"] \\
T \wedge A_0 \arrow[r, "\overline{\alpha_0}"] \arrow[d, "1\wedge w_0"'] & A_1 \arrow[d, "w_1"] \\
T \wedge B_0 \arrow[r, "\overline{\beta_0}"] & B_1
\end{tikzcd}$$

We can lift the isomorphism $w_1\overline{\varphi_1}$ and get a motivic weak equivalence $\xi_1 : C_1 \to B_1$. Then two maps $\xi_1\gamma_0$ and $\beta_0(1 \wedge \psi_0)$ from $T \wedge C_0$ to B_1 agree in the homotopy category. Therefore, there is a (left or) right homotopy between them. (See 1.2.6 and 1.2.10 [10].) In other words, there exists a path object B_1^I, (recall that a path object is a factorization of the diagonal map $B_1 \xrightarrow{r} B_1^I \xrightarrow{(p,q)} B_1 \times B_1$ with r a weak equivalence,) and a homotopy $H : T \wedge C_0 \to B_1^I$ such that $pH = \xi_1\gamma_0$ and $qH = \beta_0(1 \wedge \psi_0)$. So we have the following commutative diagram. (The dotted arrow is to be constructed.)

$$\begin{tikzcd}
C_1 \arrow[r, "\xi_1"] \arrow[dr, dotted, "\eta_1"] & B_1 \\
T \wedge C_0 \arrow[u, "\gamma_0"] \arrow[r, "H"'] \arrow[d, "1\wedge\psi_0"'] & B_1^I \arrow[u, "p"'] \arrow[d, "q"] \\
T \wedge B_0 \arrow[r, "\beta_0"'] & B_1
\end{tikzcd}$$

Note that p and q are weak equivalences and $\overline{p} = \overline{q} = \overline{r}^{-1}$ because $pr = qr = 1_{B_1}$ and r is a weak equivalence. We may assume that B_1^I is a good path object by Lemma 4.15 [4]. The consequence is that p and q are trivial fibrations by Lemma 4.14 [4]. Since γ_0 is a cofibration and p is a trivial fibration, there is a map η_1 such that $\eta_1\gamma_0 = H$ and $p\eta_1 = \xi_1$. It is a weak equivalence because so are p and ξ_1. Now define ψ_1 to be the composite $q\eta_1 : C_1 \to B_1$, which is a weak equivalence. Then the following diagram commutes: $\psi_1\gamma_0 = q\eta_1\gamma_0 = qH = \beta_0(1\wedge\psi_0)$.

$$
\begin{tikzcd}
T\wedge C_0 \arrow[r, "\gamma_0"] \arrow[d, "1\wedge\psi_0"'] & C_1 \arrow[d, "\psi_1"] \\
T\wedge B_0 \arrow[r, "\beta_0"] & B_1
\end{tikzcd}
$$

Note that $\overline{\psi_1} = \overline{q\eta_1} = \overline{p\eta_1} = \overline{\xi_1} = w_1\overline{\varphi_1}$. Thus ψ_1 is a lift of $w_1\overline{\varphi_1}$. Next, we go over to the inductive step, and assume that weak equivalences $\psi_0, \psi_1, \ldots, \psi_n$ have been constructed in such a way that ψ_i is a lift of $w_i\overline{\varphi_i}$ for $0 \leq i \leq n$, and the following diagram commutes for $0 \leq i \leq n-1$.

$$
\begin{tikzcd}
T\wedge C_i \arrow[r, "\gamma_i"] \arrow[d, "1\wedge\psi_i"'] & C_{i+1} \arrow[d, "\psi_{i+1}"] \\
T\wedge B_i \arrow[r, "\beta_i"] & B_{i+1}
\end{tikzcd}
$$

Then applying the same procedure as above to the diagram

$$
\begin{tikzcd}
T\wedge C_n \arrow[r, "\overline{\gamma_n}"] \arrow[d, "1\wedge\overline{\varphi_n}"'] \arrow[dd, bend right=60, "1\wedge\overline{\psi_n}"'] & C_{n+1} \arrow[d, "\overline{\varphi_{n+1}}"] \\
T\wedge A_n \arrow[r, "\overline{\alpha_n}"] \arrow[d, "1\wedge w_n"'] & A_{n+1} \arrow[d, "w_{n+1}"] \\
T\wedge B_n \arrow[r, "\overline{\beta_n}"] & B_{n+1}
\end{tikzcd}
$$

we can define $\psi_{n+1} : C_{n+1} \to B_{n+1}$ with similar properties as above. Thus, we get a level equivalence $\psi : C \to B$. $\square$

2.3 Smash product

The smash product $\wedge$ of (motivic) symmetric T-spectra is described in section 2 of [11], 4.3 of [13], and I.3 of [22]. We summarize the definitions here omitting the proofs of statements, which can be found in the references. We begin with standard definitions.

Definition 2.3.1 ([15, 14])**.** A *monoidal* category is a category $\mathcal{C}$ equipped with a monoidal product bifunctor $\otimes : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$, a unit object I, a natural associativity isomorphism $\alpha : (A\otimes B)\otimes C \to A\otimes(B\otimes C)$, a natural left unit isomorphism $\lambda : I\otimes A \to A$, and a natural right unit isomorphism $\rho : A \otimes I \to A$ such that the following coherence diagrams commute for all A,B,C, and D in $\mathcal{C}$.

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha\otimes 1"] \arrow[d, "\alpha"'] & (A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha"] & A \otimes ((B \otimes C) \otimes D) \arrow[d, "1\otimes\alpha"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha"'] & & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha"] \arrow[dr, "\rho\otimes 1"'] & & A \otimes (I \otimes B) \arrow[dl, "1\otimes\lambda"] \\
& A \otimes B &
\end{tikzcd}$$

A *symmetric* monoidal category is a monoidal category equipped with a natural commutativity isomorphism $\gamma : A \otimes B \to B \otimes A$ such that the following diagrams commute.

$$\begin{tikzcd}
A \otimes B \arrow[r, "\gamma"] \arrow[dr, "1"'] & B \otimes A \arrow[d, "\gamma"] \\
& A \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "a"] \arrow[d, "\gamma\otimes 1"'] & A \otimes (B \otimes C) \arrow[r, "c"] & (B \otimes C) \otimes A \arrow[d, "a"] \\
(B \otimes A) \otimes C \arrow[r, "a"'] & B \otimes (A \otimes C) \arrow[r, "1\otimes\gamma"'] & B \otimes (C \otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes A \arrow[rr, "\gamma"] \arrow[dr, "\lambda"'] & & A \otimes I \arrow[dl, "\rho"] \\
& A &
\end{tikzcd}$$

We say that a monoidal category is *closed* if the functors $A \otimes -$ and $- \otimes B$ have right adjoints.

Definition 2.3.2. Let $(\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, I_{\mathcal{D}})$ be monoidal categories. A *lax monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ together with a natural transformation $\varphi_{A,B} : FA \otimes_{\mathcal{D}} FB \to F(A \otimes_{\mathcal{C}} B)$ and a morphism $\varepsilon : I_{\mathcal{D}} \to FI_{\mathcal{C}}$ that respects the monoidal products in the sense that the following diagrams commute for all $A, B, C \in \mathcal{C}$.

$$\begin{tikzcd}
(FA \otimes_{\mathcal{D}} FB) \otimes_{\mathcal{D}} FC \arrow[r, "\varphi\otimes 1"] \arrow[d, "\alpha"'] & F(A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{D}} FC \arrow[r, "\varphi"] & F((A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C) \arrow[d, "F\alpha"] \\
FA \otimes_{\mathcal{D}} (FB \otimes_{\mathcal{D}} FC) \arrow[r, "1\otimes\varphi"'] & FA \otimes_{\mathcal{D}} F(B \otimes_{\mathcal{C}} C) \arrow[r, "\varphi"'] & F(A \otimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C))
\end{tikzcd} \tag{2.1}$$

$$
\begin{tikzcd}
I_{\mathcal{D}} \otimes_{\mathcal{D}} FA \arrow[r, "\varepsilon\otimes 1"] \arrow[d, "\lambda"'] & FI_{\mathcal{C}} \otimes_{\mathcal{D}} FA \arrow[d, "\varphi"] \\
FA & F(I_{\mathcal{C}} \otimes_{\mathcal{C}} A) \arrow[l, "F\lambda"]
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes_{\mathcal{D}} I_{\mathcal{D}} \arrow[r, "1\otimes\varepsilon"] \arrow[d, "\rho"'] & FA \otimes_{\mathcal{D}} FI_{\mathcal{C}} \arrow[d, "\varphi"] \\
FA & F(A \otimes_{\mathcal{C}} I_{\mathcal{C}}) \arrow[l, "F\rho"]
\end{tikzcd}
\tag{2.2}
$$

Definition 2.3.3. In a monoidal category $\mathcal{C}$ with monoidal product $\otimes$ and the unit object I, an object A together with two morphisms $\mu : A \otimes A \to A$ and $\eta : I \to A$ is called a *monoid* if the following diagrams commute.

$$
\begin{tikzcd}
(A \otimes A) \otimes A \arrow[r, "\alpha"] \arrow[d, "\mu\otimes 1"'] & A \otimes (A \otimes A) \arrow[r, "1\otimes\mu"] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[rr, "\mu"'] & & A
\end{tikzcd}
\tag{2.3}
$$

$$
\begin{tikzcd}
I \otimes A \arrow[r, "\eta\otimes 1"] \arrow[dr, "\lambda"'] & A \otimes A \arrow[d, "\mu"] & A \otimes I \arrow[l, "1\otimes\eta"'] \arrow[dl, "\rho"] \\
& A &
\end{tikzcd}
\tag{2.4}
$$

The morphisms μ and η are called *multiplication* and *unit*, respectively. Suppose the monoidal category $\mathcal{C}$ is symmetric with the commutativity isomorphism γ. Then a monoid A is said to be *commutative* if $\mu \circ \gamma = \mu$.

Lemma 2.3.4. *Lax monoidal functors send monoids to monoids.*

Proof. Suppose $F : \mathcal{C} \to \mathcal{D}$ is a lax monoidal functor between monoidal categories $\mathcal{C}$ and $\mathcal{D}$, and let A be a monoid in $\mathcal{C}$. The multiplication μ and the unit η of A induces the multiplication μ' and the unit η' of FA as follows.

$$\mu' : FA \otimes FA \xrightarrow{\varphi} F(A \otimes A) \xrightarrow{F\mu} FA$$

$$\eta' : I_{\mathcal{D}} \xrightarrow{\varepsilon} FI_{\mathcal{C}} \xrightarrow{F\eta} FA$$

Then using the definition of μ', we can expand the associativity diagram for FA to the following diagram, which is commutative by (2.1), (2.3), and the functoriality of φ.

$$
\begin{tikzcd}
(FA \otimes FA) \otimes FA \arrow[r, "\alpha"] \arrow[d, "\varphi\otimes 1"'] & FA \otimes (FA \otimes FA) \arrow[r, "1\otimes\varphi"] & FA \otimes F(A \otimes A) \arrow[r, "F1\otimes F\mu"] \arrow[d, "\varphi"'] & FA \otimes FA \arrow[d, "\varphi"] \\
F(A \otimes A) \otimes FA \arrow[r, "\varphi"] \arrow[d, "F\mu\otimes F1"'] & F((A \otimes A) \otimes A) \arrow[r, "F\alpha"] \arrow[d, "F(\mu\otimes 1)"] & F(A \otimes (A \otimes A)) \arrow[r, "F(1\otimes\mu)"] & F(A \otimes A) \arrow[d, "F\mu"] \\
FA \otimes FA \arrow[r, "\varphi"'] & F(A \otimes A) \arrow[rr, "F\mu"'] & & FA
\end{tikzcd}
$$

Similarly, the unit diagram for FA is expanded to the following diagram, which is commutative by (2.4), (2.2), and the functoriality of φ.

$$\begin{tikzcd}
I_{\mathcal{D}} \otimes FA \arrow[r, "\varepsilon\otimes 1"] \arrow[ddrr, bend right, "\lambda"'] & FI_{\mathcal{C}} \otimes FA \arrow[r, "F\eta\otimes 1"] \arrow[d, "\varphi"'] & FA \otimes FA \arrow[d, "\varphi"'] & FA \otimes FI_{\mathcal{C}} \arrow[l, "1\otimes F\eta"'] \arrow[d, "\varphi"] & FA \otimes I_{\mathcal{D}} \arrow[l, "1\otimes\varepsilon"'] \arrow[ddll, bend left, "\rho"] \\
& F(I_{\mathcal{C}} \otimes A) \arrow[r, "F(\eta\otimes 1)"] \arrow[dr, "F\lambda"'] & F(A \otimes A) \arrow[d] & F(A \otimes I_{\mathcal{C}}) \arrow[l, "F(1\otimes\eta)"'] \arrow[dl, "F\rho"] & \\
& & FA & &
\end{tikzcd}$$

$\square$

Definition 2.3.5 (4.2.6 [10]). A *(symmetric) monoidal model category* $\mathcal{C}$ is a closed (symmetric) monoidal category with a model structure making $\mathcal{C}$ into a model category, such that the following conditions hold.

(1) If $f : A \to B$ and $g : C \to D$ are cofibrations, then the induced map

$$(B \otimes C) \coprod_{A\otimes C} (A \otimes D) \to B \otimes D$$

is a cofibration that is trivial if either f or g is.

(2) If $q : I' \to I$ is a cofibrant replacement of the unit object I, then for all cofibrant A, the natural maps

$$I' \otimes A \xrightarrow{q\otimes 1} I \otimes A$$

$$A \otimes I' \xrightarrow{1\otimes q} A \otimes I$$

are weak equivalences.

Theorem 2.3.6 (4.3.2 [10]). *Suppose $\mathcal{C}$ is a (symmetric) monoidal model category. Then the associated homotopy category $H(\mathcal{C})$ can be given the structure of a closed (symmetric) monoidal category. The monoidal product $\otimes^L$ on $H(\mathcal{C})$ is the total left derived functor of the monoidal product $\otimes$ on $\mathcal{C}$, and the associativity and unit isomorphisms (and the commutativity isomorphism in case $\mathcal{C}$ is symmetric) on $H(\mathcal{C})$ are derived from the corresponding isomorphisms of $\mathcal{C}$.*

If we denote the cofibrant replacement functor by Q, then $\otimes^L$ is given by

$$A \otimes^L B = QA \otimes QB.$$

The associativity isomorphism, the unit isomorphisms and the commutativity isomorphism in the homotopy category are defined as follows. (Note that the condition 2 of Definition 2.3.5 is needed for the unit maps to be isomorphisms in the homotopy category.)

$$\alpha^L : Q(QA \otimes QB) \otimes QC \xrightarrow[\cong]{q\otimes 1} (QA \otimes QB) \otimes QC \xrightarrow[\cong]{\alpha} QA \otimes (QB \otimes QC) \xrightarrow[\cong]{(1\otimes q)^{-1}} QA \otimes Q(QB \otimes QC)$$

$$\lambda^L : QI \otimes QA \xrightarrow[\cong]{q\otimes 1} I \otimes QA \xrightarrow[\cong]{\lambda} QA \xrightarrow[\cong]{q} A$$

$$\rho^L : QA \otimes QI \xrightarrow[\cong]{1\otimes q} QA \otimes I \xrightarrow[\cong]{\rho} QA \xrightarrow[\cong]{q} A$$

$$\gamma^L : QA \otimes QB \xrightarrow[\cong]{\gamma} QB \otimes QA$$

Lemma 2.3.7. *The canonical functor from a monoidal model category $\mathcal{C}$ to the associated homotopy $H(\mathcal{C})$ is lax monoidal. In particular, if A is a monoid in $\mathcal{C}$, then it is a monoid in $H(\mathcal{C})$ as well.*

Proof. The natural transformation $\varphi_{A,B} : A \otimes^L B \to A \otimes B$ is defined to be $q \otimes q$ where q is the cofibrant replacement functor, and the morphism $\varepsilon : I \to I$ is defined to be the identity. The associativity diagram (2.1) translates into the following diagram, whose commutativity needs to be established.

$$\begin{tikzcd}
Q(QA \otimes QB) \otimes QC \arrow[r, "Q(q\otimes q)\otimes 1"] \arrow[d, "\alpha^L"] & Q(A \otimes B) \otimes QC \arrow[r, "q\otimes q"] & (A \otimes B) \otimes C \arrow[d, "\alpha"] \\
QA \otimes Q(QB \otimes QC) \arrow[r, "1\otimes Q(q\otimes q)"'] & QA \otimes Q(B \otimes C) \arrow[r, "q\otimes q"'] & A \otimes (B \otimes C)
\end{tikzcd} \tag{2.5}$$

Applying the cofibrant replacement functor Q to $q \otimes q : QA \otimes QB \to A \otimes B$, we get the following commutative diagram.

$$\begin{tikzcd}
Q(QA \otimes QB) \arrow[r, "Q(q\otimes q)"] \arrow[d, "q"', "\cong"] & Q(A \otimes B) \arrow[d, "q"', "\cong"] \\
QA \otimes QB \arrow[r, "q\otimes q"'] & A \otimes B
\end{tikzcd} \tag{2.6}$$

Expanding the diagram (2.5) using the definition of α^L, we get the following diagram, which is commutative by (2.6) and the functoriality of the associativity morphism α.

$$\begin{tikzcd}
Q(QA \otimes QB) \otimes QC \arrow[r, "Q(q\otimes q)\otimes 1"] \arrow[d, "q\otimes 1"'] & Q(A \otimes B) \otimes QC \arrow[d, "q\otimes 1"'] \arrow[dr, "q\otimes q"] & \\
(QA \otimes QA) \otimes QC \arrow[r, "(q\otimes q)\otimes 1"] \arrow[d, "\alpha"'] & (A \otimes B) \otimes QC \arrow[r, "1\otimes q"] & (A \otimes B) \otimes C \arrow[d, "\alpha"] \\
QA \otimes (QB \otimes QC) \arrow[r, "1\otimes(q\otimes q)"] \arrow[d, "(1\otimes q)^{-1}"'] & QA \otimes (B \otimes C) \arrow[r, "q\otimes 1"] \arrow[d, "(1\otimes q)^{-1}"'] & A \otimes (B \otimes C) \\
QA \otimes Q(QB \otimes QC) \arrow[r, "1\otimes Q(q\otimes q)"'] & QA \otimes Q(B \otimes C) \arrow[ur, "q\otimes q"'] &
\end{tikzcd}$$

By the definition of ε and φ, to prove the commutativity of the diagram (2.2) is to prove $\lambda(q \otimes q) = \lambda^L$ and $\rho(q \otimes q) = \rho^L$. The following diagram, which is commutative by the functoriality of the isomorphism λ, shows the first equality since $\lambda^L = q\lambda(q \otimes 1)$. The second equality is shown similarly.

$$\begin{tikzcd}
QI \otimes QA \arrow[r, "q\otimes 1"] \arrow[dr, "q\otimes q"'] & I \otimes QA \arrow[r, "\lambda"] \arrow[d, "1\otimes q"] & QA \arrow[d, "q"] \\
 & I \otimes A \arrow[r, "\lambda"'] & A
\end{tikzcd}$$

□

A *symmetric sequence* is a sequence of motivic spaces $A_0, A_1, \ldots, A_n, \ldots$ with a base point preserving left action of Σ_n on A_n for each $n \geq 0$. The underlying spaces of a motivic symmetric spectrum form a symmetric sequence. There is an obvious category of symmetric sequences denoted by $\mathbf{M}_\bullet(S)^\Sigma$ following the practice in [11]. The *tensor product* of symmetric sequences A and B is defined to be the symmetric sequence

$$(A \otimes B)_n = \bigvee_{p+q=n} (\Sigma_n)_+ \wedge_{\Sigma_p \times \Sigma_q} (A_p \wedge B_q).$$

Then a map $A \otimes B \to C$ of symmetric sequences is characterized by a collection of $\Sigma_p \times \Sigma_q$-equivariant maps $A_p \wedge B_q \to C_{p+q}$. There is a *twist isomorphism* $\gamma : A \otimes B \to B \otimes A$ determined by

$$A_p \wedge B_q \xrightarrow{t} B_q \wedge A_p \xrightarrow{in} (B \otimes A)_{q+p} \xrightarrow{c_{q,p}} (B \otimes A)_{p+q}$$

where t is the twist of smash factors, in is the inclusion, and $c_{q,p} \in \Sigma_{p+q}$ is the (q,p)-shuffle given by $c_{q,p}(i) = i + p$ for $1 \leq i \leq q$ and $c_{q,p}(i) = i - q$ for $q + 1 \leq i \leq q + p$.

Theorem 2.3.8 ([11])**.** *The category $\mathbf{M}_{\bullet}(S)^{\Sigma}$ with the tensor product $\otimes$ and the twist isomorphism γ is a closed symmetric monoidal category.*

Proof. See Lemma 2.1.6 of [11] for the proof that the category is symmetric monoidal. The argument works when the category of pointed simplicial sets is replaced by the category of pointed motivic spaces. The existence of internal hom object follows formally from the argument of Theorem 2.1.11 of [11] if the simplicial set of maps $\mathrm{Map}(-,-)$ (loc. cit.) is replaced by the internal hom object $\mathbf{Hom}_{\bullet}(-,-)$ of the category of motivic spaces □

In this category, the symmetric sequence $\mathcal{T} = (T^0, T^1, T^2, \ldots, T^n, \ldots)$ is a commutative monoid, that is $\mu \circ \gamma = \mu$ where γ is the twist isomorphism and $\mu : \mathcal{T} \otimes \mathcal{T} \to \mathcal{T}$ is the map determined by the obvious maps $T^p \wedge T^q \to T^{p+q}$. A motivic symmetric T-spectrum A is a (left) $\mathcal{T}$-module in the sense that the iterated structure maps σ^p determine a map of symmetric sequences

$$m : \mathcal{T} \otimes A \to A.$$

that satisfies the usual associativity condition.

The *smash product* of symmetric spectra A and B is defined to be the symmetric sequence coequalizer

$$\mathcal{T} \otimes A \otimes B \rightrightarrows A \otimes B \to A \wedge B$$

of the maps $m \otimes 1$ and the composite

$$\mathcal{T} \otimes A \otimes B \xrightarrow{\gamma \otimes 1} A \otimes \mathcal{T} \otimes B \xrightarrow{1 \otimes m} A \otimes B.$$

The $\mathcal{T}$-module structure on $A \wedge B$ is induced by the map $m \otimes 1 : \mathcal{T} \otimes A \otimes B \to A \otimes B$. Then a map $A \wedge B \to C$ of motivic symmetric spectra is characterized by $\Sigma_p \times \Sigma_q$-equivariant maps

$$h_{p,q} : A_p \wedge B_q \to C_{p+q}, \quad p, q \geq 0$$

such that the following diagrams commute for each $p, q, r \geq 0$.

$$\begin{tikzcd}
T^r \wedge A_p \wedge B_q \arrow[r, "\sigma \wedge 1"] \arrow[d, "1 \wedge h_{p,q}"'] & A_{r+p} \wedge B_q \arrow[d, "h_{r+p,q}"] \\
T^r \wedge C_{p+q} \arrow[r, "\sigma"'] & C_{r+p+q}
\end{tikzcd}$$

$$\begin{tikzcd}
T^r \wedge A_p \wedge B_q \arrow[r, "t\wedge 1"] \arrow[d, "\sigma\wedge 1"'] & A_p \wedge T^r \wedge B_q \arrow[r, "1\wedge\sigma"] & A_p \wedge B_{r+q} \arrow[d, "h_{p,r+q}"] \\
A_{r+p} \wedge B_q \arrow[r, "h_{r+p,q}"'] & C_{r+p+q} \arrow[r, "\theta"'] & C_{p+r+q}
\end{tikzcd}$$

In the diagram above, $\theta \in \Sigma_{p+r+q}$ is the (r,p)-shuffle given by $\theta(i) = i + p$ for $1 \leq i \leq r$, $\theta(i) = i - r$ for $r + 1 \leq i \leq r + p$, and $\theta(i) = i$ for $r + p + 1 \leq i \leq r + p + q$.

The next two lemmas are used to show that $\mathbf{SM}^{\Sigma}(S)$ is a closed symmetric monoidal category.

Lemma 2.3.9 (2.2.2 [11])**.** *Let $\mathcal{C}$ be a symmetric monoidal category that is cocomplete and let R be a commutative monoid in $\mathcal{C}$ such that the functor*

$$R \otimes - : \mathcal{C} \to \mathcal{C}$$

preserves coequalizers. Then there is a symmetric monoidal product $\otimes_R$ on the category of R-modules with R as the unit.

Lemma 2.3.10 (2.2.8 [11])**.** *Let $\mathcal{C}$ be a closed symmetric monoidal category that is bicomplete and let R be a commutative monoid in $\mathcal{C}$. Then there is a function R-module* $\mathrm{Hom}_R(M, N)$*, natural for $M, N \in \mathcal{C}$, such that the functor $- \otimes_R M$ is left adjoint to the functor* $\mathrm{Hom}_R(M, -)$*.*

In the context of the present text, R is $\mathcal{T}$ and $\otimes_R$ is the smash product $\wedge$ defined above.

Theorem 2.3.11 (Jardine)**.** *The category of motivic symmetric spectra $\mathbf{SM}^{\Sigma}(S)$ together with the stable model category structure and the smash product defined above is a symmetric monoidal model category. It induces a symmetric monoidal category structure on $\mathbf{SH}^{\Sigma}(S) = H(\mathbf{SM}^{\Sigma}(S))$.*

Proof. It follows from Theorem 2.3.8 and Lemma 2.3.9 that $\mathbf{SM}^{\Sigma}(S)$ with the smash product is a symmetric monoidal category. It depends on the fact that $\mathcal{T}$ is a commutative monoid with respect to $\otimes$. The monoidal structure is closed by Lemma 2.3.10 and Theorem 2.3.8. To prove that $\mathbf{SM}^{\Sigma}(S)$ is a symmetric monoidal model category, we need to verify two conditions of Definition 2.3.5. See Proposition 4.19 of [13] for the first condition. The second condition is satisfied since the unit object $\mathcal{T}$ is cofibrant. Finally, the symmetric monoidal structure of $\mathbf{SM}^{\Sigma}(S)$ induces that of $H(\mathbf{SM}^{\Sigma}(S))$ by Theorem 2.3.6. □

Definition 2.3.12. A monoid in $\mathbf{SM}^{\Sigma}(S)$ is called a *motivic symmetric ring spectrum*, and a monoid in $H(\mathbf{SM}^{\Sigma}(S))$ is called a *motivic homotopy symmetric ring spectrum.*

A motivic symmetric ring spectrum induces a motivic homotopy symmetric ring spectrum by Lemma 2.3.7, whose converse is not always true. For example, the topological Moore spectrum for an odd prime

$p \geq 5$ is a homotopy ring spectrum, but not a ring spectrum. In [20, 2.2.1], Panin, Pimenov, and Röndigs supplies a motivic homotopy ring spectrum representing algebraic K-theory, but not an explicit motivic symmetric spectrum. In this paper, we show that the K-theory spectrum constructed in section 5 is a motivic symmetric ring spectrum.

2.4 Base change

Let $f : S' \to S$ be a map of base schemes. It induces the pullback functor $f^{-1} : Sm/S \to Sm/S'$ defined by $f^{-1}X = S' \times_S X$. Let $f_* : \mathbf{M}(S') \to \mathbf{M}(S)$ be the functor defined by the composition with $(f^{-1})^{op}$. It sends $B \in \mathbf{M}(S')$ to the motivic space $X \mapsto B(S' \times_S X)$. It is a general fact on the category of presheaves that f_* has a left adjoint $f^* : \mathbf{M}(S) \to \mathbf{M}(S')$ (See [1, I.5]). For $A \in \mathbf{M}(S)$ and $X' \in Sm/S'$, $(f^*A)(X')$ is defined to be $\varinjlim A(X)$ where the limit is taken over the category whose objects are pairs (X, m) where $X \in Sm/S$ and $m : X' \to S' \times_S X$, and whose morphisms are maps $g : X_1 \to X_2$ over S such that the following diagram commutes.

$$\begin{tikzcd} X' \arrow[r, "m_1"] \arrow[dr, "m_2"'] & S' \times_S X_1 \arrow[d, "1\times g"] \\ & S' \times_S X_2 \end{tikzcd}$$

Note that if f is smooth, then X' may be considered as an object of Sm/S, and $(f^*A)(X')$ is naturally isomorphic to $A(X')$. The same definition works for pointed motivic spaces and, it respects group actions. If G is a group, let $\mathbf{M}_\bullet(S)^G$ denote the category of pointed motivic spaces with left G-actions. The morphisms are base point preserving G-equivariant maps.

Proposition 2.4.1. *Suppose $f : S' \to S$ is a map of base schemes, and G is a group. Then there is an adjoint pair $(f^*, f_*) : \mathbf{M}_\bullet(S)^G \to \mathbf{M}_\bullet(S')^G$ defined as above.*

$$\mathrm{Hom}_{\mathbf{M}_\bullet(S')^G}(f^*(A), B) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{M}_\bullet(S)^G}(A, f_*(B))$$

Lemma 2.4.2. *Suppose $f : S' \to S$ is a map of schemes, and G is a group. Then*

(1) $f_*(A \wedge B) = f_*(A) \wedge f_*(B)$ for $A, B \in \mathbf{M}_\bullet(S')^G$.

(2) $f^*(A \wedge B) \cong f^*(A) \wedge f^*(B)$ for $A, B \in \mathbf{M}_\bullet(S)^G$. The isomorphism is functorial in both variables.

Proof. The first one follows from the definition. The second one is true since $\varinjlim$ of simplicial sets commutes with finite products and colimits. ☐

Proposition 2.4.3. *Suppose $f : S' \to S$ is a map of base schemes, and A is a motivic symmetric T-spectrum. Then f induces a motivic symmetric f^*T-spectrum f^*A.*

Proof. The n-th space of f^*A is defined to be f^*A_n. The structure maps of f^*A are derived from those of A and Lemma 2.4.2.

$$\sigma' : f^*T \wedge f^*A_n \cong f^*(T \wedge A_n) \xrightarrow{f^*\sigma} f^*A_{1+n}$$

The iterated structure map $(\sigma')^p : (f^*T)^p \wedge f^*A_n \to f^*A_{p+n}$ is $\Sigma_p \times \Sigma_n$-equivariant since it is isomorphic to the composite $(f^*T)^p \wedge f^*A_n \xrightarrow{\cong} f^*(T^p \wedge A_n) \xrightarrow{f^*\sigma^p} f^*A_{p+n}$, which is a $\Sigma_p \times \Sigma_n$-equivariant map by Proposition 2.4.1. $\square$

Chapter 3

Vector bundles

This chapter begins with the review of Grothendieck topology from [2, 5]. Most known definitions, theorems and ideas of proofs are from chapter 2 of [5]. Then the category of *standard vector bundles* (Definition 3.3.1) is constructed. It is a category that is equivalent to the category of locally free sheaves of finite rank with the properties listed in Theorem 3.3.8 and Theorem 3.3.10. Those properties are useful in solving technical difficulties in the construction of motivic symmetric spectrum $\mathcal{K}$ representing algebraic K-theory in chapter 5. The construction of standard vector bundles uses the idea of big vector bundles [6, C.4] originally from Grayson [9, p.169] and incorporates the new concept of presheaves on sieves. This construction also solves the question posed in [6, p.846], the existence of strictly functorial tensor product for vector bundles, and makes what they called small vector bundles redundant.

3.1 Grothendieck topology and sieves

Suppose $\mathbb{T}$ is a small category with all fibered products. A Grothendieck topology on $\mathbb{T}$ is an assignment to each object U of a collection of sets of morphisms $\{U_i \to U\}$ called coverings of U such that the following conditions are satisfied.

(1) If $V \to U$ is an isomorphism, then $\{V \to U\}$ is a covering.

(2) If $\{U_i \to U\}$ is a covering and $V \to U$ is an arrow, then $\{U_i \times_U V \to V\}$ is a covering.

(3) If $\{U_i \to U\}$ is a covering and for all i, $\{U_{ij} \to U_i\}$ is a covering, then the collection of composites $\{U_{ij} \to U_i \to U\}$ is a covering.

A category with a Grothendieck topology is called a *site*. In this chapter, we mainly use Zariski sites on a scheme X. The small Zariski site on a scheme X is the category X_{zar} whose objects are open immersions $U \to X$ and arrows are the open immersions $V \to U$ compatible with the maps to X. A covering on U is a collection of open immersions $\{f_i : U_i \to U\}$ such that $\bigcup_i f_i(U_i) = U$. The big Zariski site $(Sch/X)_{Zar}$ on

X is the category Sch/X where a covering on an object Y is a collection of open immersions $\{g_i : V_i \to Y\}$ such that $\bigcup_i g_i(V_i) = Y$.

A *sieve* on an object U of $\mathbb{T}$ is a subfunctor of the representable functor $H_U = \mathrm{Hom}_{\mathbb{T}}(-, U)$. Given a sieve H on U, we can associate a full subcategory $\mathcal{C}_H$ of the comma category $\mathbb{T}/U$ over U whose objects are the elements of $H(V)$ where V runs over the objects of $\mathbb{T}$. For simpler notations, when we refer to an object $V \xrightarrow{f} U$ of $\mathcal{C}_H$, we will frequently suppress the structure map and simply write V. No confusion should arise unless two structure maps are needed from the same object. The category $\mathcal{C}_H$ satisfies the following property.

(I) If V is an object of $\mathcal{C}_H$ and $W \to V$ is any arrow in $\mathbb{T}$, then the composite $W \to V \to U$ is also an object of $\mathcal{C}_H$.

Conversely, given a full subcategory of $\mathbb{T}/U$ satisfying the property, we can recover the subfunctor H by defining $H(V)$ to be the collection of arrows $V \to U$ in the category. Note that the property (I) above implies that the intersection of two sieves is also a sieve.

Given a collection of morphisms $\mathcal{U} = \{U_i \to U\}$, we associate a sieve $H_{\mathcal{U}}$ on U by taking

$$H_{\mathcal{U}}(V) = \{f : V \to U \mid f \text{ factors through } U_i \to U \text{ for some } i\}$$

If $\mathbb{T}$ is a site, then a sieve H is said to *belong to* $\mathbb{T}$ if H contains a sieve $H_{\mathcal{U}}$ associated to some covering $\mathcal{U}$ of U in $\mathbb{T}$. It is equivalent to say that $\mathcal{C}_H$ contains $\mathcal{U}$.

A covering $\mathcal{V} = \{V_j \to U\}$ is said to be a *refinement* of $\mathcal{U} = \{U_i \to U\}$ if every arrow $V_j \to U$ factors through $U_i \to U$ for some i. The condition is equivalent to $H_{\mathcal{V}} \subseteq H_{\mathcal{U}}$. If $\mathcal{U}_1 = \{U_{1i} \to U\}$ and $\mathcal{U}_2 = \{U_{2j} \to U\}$ are coverings of U, then let $\mathcal{U}_1 \times \mathcal{U}_2 = \{U_{1i} \times_U U_{2j} \to U\}$. It is a covering of U and is a common refinement of $\mathcal{U}_1$ and $\mathcal{U}_2$.

Proposition 3.1.1 (2.44 [5]). *If H_1 and H_2 are sieves on U belonging to $\mathbb{T}$, then the intersection $H_1 \cap H_2$ also belongs to $\mathbb{T}$.*

Proof. Let $\mathcal{U}_1 = \{U_{1i} \to U\}$ and $\mathcal{U}_2 = \{U_{2j} \to U\}$ be coverings such that $H_{\mathcal{U}_1} \subseteq H_1$ and $H_{\mathcal{U}_2} \subseteq H_2$. Then $H_1 \cap H_2$ contains $H_{\mathcal{U}_1 \times \mathcal{U}_2}$ □

Suppose $f : Y \to X$ is a map of schemes, and consider big Zariski sites $(Sch/X)_{Zar}$ and $(Sch/Y)_{Zar}$. If $V \in (Sch/Y)_{Zar}$, $U \in (Sch/X)_{Zar}$ and $g : V \to U$ is a map of schemes such that the following diagram commutes,

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "b"'] & U \arrow[d, "a"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

then for any sieve H on U the pullback g^*H is defined as a sieve on V. For each $W \in (Sch/Y)_{Zar}$, which is also an object of $(Sch/X)_{Zar}$ via f, the set $g^*H(W)$ is defined to be the set of all maps $W \to V$ such that its composition with g is an element of $H(W)$.

Proposition 3.1.2. *Suppose $f : Y \to X$ is a map of schemes. If $U \xrightarrow{a} X$ is in Sch/X, $V \xrightarrow{b} Y$ is in Sch/Y, $g : V \to U$ is a map of schemes such that $ag = fb$, and H is a sieve on U belonging to $(Sch/X)_{Zar}$, then g^*H is a sieve on V belonging to $(Sch/Y)_{Zar}$.*

Proof. Suppose H contains $H_{\mathcal{U}}$ where $\mathcal{U}$ is a Zariski covering of U, then g^*H contains $H_{g^*\mathcal{U}}$ where $g^*\mathcal{U} = \{U_i \times_U V \to V\}$, which is a Zariski covering of V. □

3.2 Presheaves and sheaves

Definition 3.2.1. Let X be an object of a site $\mathbb{T}$ and suppose H is a sieve on X belonging to the site.

(1) An H-*presheaf* is a functor $\mathcal{C}_H^{op} \to \mathbf{Set}$.

(2) An H-*sheaf* is an H-presheaf F such that for each object U of $\mathcal{C}_H$ and a covering $\{U_i \to U\}$, the diagram

$$F(U) \longrightarrow \prod F(U_i) \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} \prod F(U_i \times_U U_j)$$

is exact where p_1 and p_2 are projections to the first and the second factors of $U_i \times_U U_j$.

(3) An H-presheaf F is said to be *separated* if for each object U of $\mathcal{C}_H$ and a covering $\{U_i \to U\}$, the map $F(U) \to \prod F(U_i)$ is injective.

Here we use the convention that the value of F on an object $U \xrightarrow{f} X$ of $\mathcal{C}_H$ is written as $F(U)$ assuming that the structure map f is understood. When we need to consider two different structure maps f and g, we will distinguish them by writing $F(U_{[f]})$ and $F(U_{[g]})$. By replacing the category of sets by the category of abelian groups, rings, etc., we get the definitions of H-presheaves of abelian groups, rings, etc. A map of H-presheaves, (H-sheaves, separated H-presheaves) is a natural transformation of functors. We denote the category of H-presheaves, H-sheaves, and separated H-presheaves by $Pre_H(\mathbb{T})$, $Shv_H(\mathbb{T})$, and $Pre_H^s(\mathbb{T})$, respectively. Then $Shv_H(\mathbb{T}) \subseteq Pre_H^s(\mathbb{T}) \subseteq Pre_H(\mathbb{T})$. Suppose H and K are sieves belonging to $\mathbb{T}$ and

$K \subseteq H$. Then $\mathcal{C}_K$ is a full subcategory of $\mathcal{C}_H$, and the composition with the inclusion functor induces functors $Pre_H(\mathbb{T}) \to Pre_K(\mathbb{T})$, $Shv_H(\mathbb{T}) \to Shv_K(\mathbb{T})$, and $Pre^s_H(\mathbb{T}) \to Pre^s_K(\mathbb{T})$ called restrictions. These functors will be denoted by $-|_K$ universally. Intuitively, we may consider an H-(pre)sheaf as a (pre)sheaf defined only on *small* open sets. If the site $\mathbb{T}$ has a final object X and $H = \mathrm{Hom}_{\mathbb{T}}(-, X)$, the biggest sieve on X, then $\mathcal{C}_H = \mathbb{T}$. In this case an H-presheaf (H-sheaf, or separated H-presheaf) is a presheaf (sheaf, or separated presheaf) in the usual sense, and we write the category of presheaves, separated presheaves, and sheaves as $Pre(\mathbb{T})$, $Pre^s(\mathbb{T})$, and $Shv(\mathbb{T})$, respectively.

In the next theorem, we sheafify an H-presheaf to obtain a sheaf. Only local information is needed to define a sheaf after all. The construction is the same as the construction of the sheafification of a presheaf in the usual sense, only uses less but sufficient information. The proof mimics the proof of Theorem 2.64 in [5] that shows the existence of the sheafification of a presheaf in the usual sense. In its proof, *locally equal* sections are identified to get a separated presheaf, then *locally defined* sections are patched together to obtain a sheaf. The proof works for sheaves of abelian groups, rings, etc., too.

Theorem 3.2.2. *Let $\mathbb{T}$ be a site, X a final object of $\mathbb{T}$, H a sieve on X belonging to $\mathbb{T}$ and $\eta : Shv(\mathbb{T}) \to Pre_H(\mathbb{T})$ the restriction functor. Then there is a functor $\xi_H : Pre_H(\mathbb{T}) \to Shv(\mathbb{T})$ called sheafification and a natural bijection*

$$\mathrm{Hom}_{Shv(\mathbb{T})}(\xi_H F, G) \cong \mathrm{Hom}_{Pre_H(\mathbb{T})}(F, \eta G)$$

Proof. For simpler notation, we will use ξ for ξ_H. Suppose F is an H-presheaf. We construct a separated H-presheaf F^s by taking $F^s(U) = F(U)/\sim$ where we say $s \sim t$ for $s, t \in F(U)$ if there is a covering $\{U_i \to U\}$ such that the pullbacks of s and t to each U_i coincide. If $V \to U$ is an arrow in $\mathcal{C}_H$, the pullback $F(U) \to F(V)$ is compatible with the equivalence relation, so we have a map $F^s(U) \to F^s(V)$. We have a surjective natural transformation $F \to F^s$ such that every map from F to a separated H-presheaf factors through F^s. In particular, if $\gamma : F \to G$ is a map of H-presheaves, we get a map $\gamma^s : F^s \to G^s$.

Next, we construct a sheaf from F^s. Suppose U is an object of $\mathbb{T}$. Consider the set of pairs $(\{U_i \to U\}, \{s_i\})$ where $\{U_i \to U\}$ is a covering of U such that each U_i is in $\mathcal{C}_H$, $s_i \in F^s(U_i)$, and the pullbacks of s_i and s_j to $U_i \times_U U_j$ coincide. We declare $(\{U_i \to U\}, \{s_i\})$ and $(\{V_j \to U\}, \{t_j\})$ are equivalent if the pullbacks of s_i and t_j to $U_i \times_U V_j$ coincide. The relation is transitive since F^s is separated. We let $\xi F(U)$ be the set of equivalence classes. Note that $(\{U_i \to U\}, \{s_i\})$ is equivalent to $(\{V_i \to U\}, \{t_i\})$ if there is an isomorphism $f_i : V_i \to U_i$ over U for each i and $t_i = f_i^* s_i$. Given a map $g : V \to U$, we send the class of $(\{U_i \to U\}, \{s_i\})$ to the class of $(\{U_i \times_U V \to V\}, \{p_i^* s_i\})$ where $p_i^* s_i$ is the pullback of s_i along the map $p_i : U_i \times_U V \to U_i$. The definition does not depend on the representative of the class because if two sections coincide in $F^s(U_i \times_U V_j)$, then their pullbacks coincide in $F^s\big((U_i \times_U V) \times_V (V_j \times_U V)\big) \cong F^s\big((U_i \times_U V_j) \times_U (V \times_V V)\big)$. This defines a map $\xi F(g) : \xi F(U) \to \xi F(V)$. If $h : W \to V$ is another map in $\mathcal{C}_H$, then $\xi F(gh) = \xi F(h)\xi F(g)$ because if we let q_i be the projection $(U_i \times_U V) \times_V W \to U_i \times_U V$, and r_i the projection $U_i \times_U W \to U_i$, then the pairs $(\{(U_i \times_U V) \times_V W \to W\}, \{q_i^* p_i^* s_i\})$ and $(\{U_i \times_U W \to W\}, \{r_i^* s_i\})$ are equivalent. Hence ξF is a presheaf.

Now we show that ξF satisfies the sheaf conditions. Let $\{U_i \to U\}_{i\in I}$ be a covering. Consider the following sections:

$$([\sigma_i])_{i\in I} = ([\{U_{ik} \to U_i\}_{k\in K_i}, \{s_{ik}\}_{k\in K_i}])_{i\in I} \in \prod_{i\in I} \xi F(U_i).$$

Assume that the pullbacks of $[\sigma_i]$ and $[\sigma_j]$ in $\xi F(U_i \times_U U_j)$ coincide, which means that for each $i, j \in I$, $(\{U_{ik} \times_U U_j \to U_i \times_U U_j\}, \{p_{ik}^* s_{ik}\})$ is equivalent to $(\{U_i \times_U U_{jl} \to U_i \times_U U_j\}, \{q_{jl}^* s_{jl}\})$ where p_{ik} and q_{jl} are projections $U_{ik} \times_U U_j \to U_{ik}$ and $U_i \times_U U_{jl} \to U_{jl}$. Then the pullbacks of s_{ik} and s_{jl} in $F^s(U_{ik} \times_U U_{jl})$ along those projections coincide for all $i, j \in I, k \in K_i$, and $l \in K_j$ since $U_{ik} \times_U U_{jl} \cong (U_{ik} \times_U U_j) \times_{(U_i \times_U U_j)} (U_i \times_U U_{jl})$. Therefore, the pair $\sigma = (\{U_{ik} \to U\}_{i\in I, k\in K_i}, \{s_{ik}\}_{i\in I, k\in K_i})$ defines a section in $\xi F(U)$. Now we show that the pullback of $[\sigma]$ to each U_j is $[\sigma_j]$. The pullback of $[\sigma]$ in $\xi F(U_j)$ is the class of the pair $(\{U_{ik} \times_U U_j \to U_j\}, \{p_{ik}^* s_{ik}\})$. This pair is equivalent to the pair $\sigma_j = (\{U_{jl} \to U_j\}, \{s_{jl}\})$ because the pullbacks of $p_{ik}^* s_{ik}$ and s_{jl} coincide in $F^s(U_{ik} \times_U U_{jl}) \cong F^s((U_{ik} \times_U U_j) \times_{U_j} U_{jl})$. This shows the existence. For uniqueness, suppose $\tau = (\{V_j \to U\}, \{t_j\})$ is another section of $\xi F(U)$ whose pullback in $\xi F(U_i)$ is equivalent to σ_i for all i, then the pullbacks of t_j and s_{ik} coincide in $F^s(V_j \times_U U_{ik})$ for all i, j, and k. This implies that τ is equivalent to σ. This completes the proof that ξF is a sheaf.

Now we define ξ on morphisms. Suppose $\gamma : F_1 \to F_2$ is a map of H-presheaves. For each object U of $\mathbb{T}$, define $\xi\gamma(U) : \xi F_1(U) \to \xi F_2(U)$ by sending the class of $(\{U_i \to U\}, \{s_i\})$ to $(\{U_i \to U\}, \{\gamma^s s_i\})$. If $f : V \to U$ is an arrow, the diagram

$$\begin{tikzcd}
\xi F_1(U) \arrow[r, "\xi\gamma(U)"] \arrow[d, "f^*"'] & \xi F_2(U) \arrow[d, "f^*"] \\
\xi F_1(V) \arrow[r, "\xi\gamma(V)"'] & \xi F_2(V)
\end{tikzcd}$$

commutes:

$$\begin{aligned} f^*\xi\gamma(U)[\{U_i \to U\},\{s_i\}] &= f^*[\{U_i \to U\},\{\gamma^s s_i\}] \\ &= [\{U_i \times_U V \to V\},\{p_i^*\gamma^s s_i\}] \\ &= [\{U_i \times_U V \to V\},\{\gamma^s p_i^* s_i\}] \\ &= \xi\gamma(V)[\{U_i \times_U V \to V\},\{p_i^* s_i\}] \\ &= \xi\gamma(V)f^*[\{U_i \to U\},\{s_i\}]. \end{aligned}$$

Hence $\xi\gamma$ is a map of sheaves. If $\delta : F_2 \to F_3$, is another map of H-presheaves, then it is immediate that $\xi(\delta\gamma) = (\xi\delta)(\xi\gamma)$ by definition. Therefore ξ is a functor $Pre_H(\mathbb{T}) \to Shv(\mathbb{T})$.

Next we prove that (ξ, η) is an adjoint pair. Suppose F is a H-presheaf and G is a sheaf. Given a map $\alpha : \xi F \to G$ of sheaves, define a map $\beta_\alpha : F \to \eta G$ of H-presheaves as follows. If U is an object of $\mathcal{C}_H$, $(\{U \xrightarrow{1} U\}, \bar{s})$ defines a section in $\xi F(U)$ for each $s \in F(U)$ where $\bar{s}$ is the class of s in $F^s(U)$. Define $\beta_\alpha(U) : F(U) \to \eta G(U)$ by sending s to $\alpha(U)[U \to U, \bar{s}]$. If $f : V \to U$ is an arrow of $\mathcal{C}_H$, then for all $s \in F(U)$,

$$\begin{aligned} f^*\beta_\alpha(U)(s) &= f^*\alpha(U)[\{U \to U\}, \bar{s}] \\ &= \alpha(U)f^*[U \to U, \bar{s}] \\ &= \alpha(V)[U \times_U V \to V, p^*\bar{s}] \\ &= \alpha(V)[V \to V, \overline{f^*s}] \\ &= \beta_\alpha(V)f^*s \end{aligned}$$

Therefore β_α is a map of sheaves. Conversely, given a map $\beta : F \to \eta G$ of H-presheaves, define a map $\alpha_\beta : \xi F \to G$ of sheaves as follows. A section in $\xi F(U)$ is a class of a pair $\sigma = (\{U_i \to U\}, \{s_i\})$ such that U_i is an object of $\mathcal{C}_H$ and the pullbacks of s_i and s_j to $U_i \times_U U_j$ coincide. So we have sections $\{\beta(U_i)s_i\} \in \prod \eta G(U_i)$ such that the pullbacks to $U_i \times_U U_j$ of $\beta(U_i)s_i$ and $\beta(U_j)s_j$ coincide. The sheaf condition of G determine a unique section in $G(U)$ whose pullback in each $G(U_i) = \eta G(U_i)$ is $\beta(U_i)s_i$. We call it $\alpha_\beta(U)[\sigma]$. If $\tau = (\{V_j \to U\}, \{t_j\})$ is equivalent to σ, then the pullbacks of s_i and t_j in $F(U_i \times_U V_j)$ coincide so that the pullbacks of $\alpha_\beta(U)[\sigma]$ and $\alpha_\beta(U)[\tau]$ in $G(U_i \times_U V_j)$ coincide. Therefore $\alpha_\beta(U)$ is well-defined. If $f : V \to U$ is an arrow in $\mathbb{T}$, then for each $\sigma = (\{U_i \to U\}, \{s_i\})$, the pullbacks of $f^*\alpha_\beta(U)[\sigma]$ and $\alpha_\beta(V)f^*[\sigma]$ in $G(U_i \times_U V)$ for each i are $p_i^*\beta(U_i)s_i = \beta(U_i \times_U V)p_i^*s_i$ where p_i is the projection $U_i \times_U V \to U_i$. Therefore $f^*\alpha_\beta = \alpha_\beta f^*$ and it proves that α_β is map of sheaves. For every object $U \to X$ in $\mathcal{C}_H$ and $s \in F(U)$, $\beta_{\alpha_\beta}(U)s = \alpha_\beta(U)[U \to U, s] = \beta(U)s$. Hence $\beta_{\alpha_\beta} = \beta$. For every object $U \to X$ in $\mathbb{T}$ and $[\sigma] \in \xi F(U)$, $\sigma = (\{U_i \to U\}, \{s_i\})$, The pullbacks of $\alpha_{\beta_\alpha}(U)[\sigma]$ and $\alpha(U)[\sigma]$ in $G(U_j)$ for every j are $\beta_\alpha(U_j)(s_j)$ and $\alpha(U_j)[U_i \times_U U_j \to U_j, p_i^*s_i]$. Both are equal to $\alpha(U_j)[U_j \to U_j, s_j]$. Therefore $\alpha_{\beta_\alpha} = \alpha$. This proves the bijection $\mathrm{Hom}_H(\xi F, G) \cong \mathrm{Hom}_K(F, \eta G)$.

Finally, to prove that the bijective correspondence is natural, we show that the following diagrams commute for any $\gamma : F_1 \to F_2$ and $\delta : G_1 \to G_2$.

$$\begin{tikzcd}
\mathrm{Hom}_H(\xi F_2, G) \arrow[r, "\beta."] \arrow[d, "(\xi\gamma)^*"'] & \mathrm{Hom}_K(F_2, \eta G) \arrow[d, "\gamma^*"] \\
\mathrm{Hom}_H(\xi F_1, G) \arrow[r, "\beta."'] & \mathrm{Hom}_K(F_1, \eta G)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hom}_H(\xi F, G_1) \arrow[r, "\beta."] \arrow[d, "\delta_*"'] & \mathrm{Hom}_K(F, \eta G_1) \arrow[d, "(\eta\delta)_*"] \\
\mathrm{Hom}_H(\xi F, G_2) \arrow[r, "\beta."'] & \mathrm{Hom}_K(F, \eta G_2)
\end{tikzcd}$$

If U is an object of $\mathcal{C}_H$, $s \in F_1(U)$, and $\alpha : \xi F_2 \to G$, then

$$\begin{aligned}
\gamma^* \beta_\alpha(U)s &= \beta_\alpha(U)(\gamma(U)s) \\
&= \alpha(U)[U \to U, \overline{\gamma(U)s}], \\
\beta_{(\xi\gamma)^*\alpha}(U)s &= \beta_{\alpha\xi\gamma}(U)s \\
&= (\alpha\xi\gamma)(U)[U \to U, \bar{s}] \\
&= \alpha(U)(\xi\gamma)(U)[U \to U, \bar{s}] \\
&= \alpha(U)[U \to U, \overline{\gamma(U)s}].
\end{aligned}$$

So the first diagram commutes. For the second diagram, let U be an object of $\mathcal{C}_H$, $s \in F(U)$, and $\alpha : \xi F \to G_1$. Then

$$\begin{aligned}
(\eta\delta)_* \beta_\alpha(U)s &= (\eta\delta)(U)\beta_\alpha(U)s \\
&= (\eta\delta)(U)\alpha(U)[U \to U, \bar{s}] \\
&= \delta(U)\alpha(U)[U \to U, \bar{s}] \\
&= (\delta\alpha)(U)[U \to U, \bar{s}] \\
&= \beta_{\delta\alpha}(U)s \\
&= \beta_{\delta_*\alpha}(U)s.
\end{aligned}$$

$\square$

Lemma 3.2.3. *Under the hypothesis of Theorem 3.2.2, the unit map $\epsilon : E \to (\xi_H E)|_H$ of the adjunction is an isomorphism if E is an H-sheaf.*

Proof. Since E is separated, E is identified with E^s. Using the notation of the proof of Theorem 3.2.2, for each $U \in \mathcal{C}_H$, $\epsilon(U)$ is defined by $s \mapsto [\{U \xrightarrow{1} U\}, s]$. If $s, t \in E(U)$, and $[\{U \xrightarrow{1} U\}, s] = [\{U \xrightarrow{1} U\}, t]$, then there is a covering $\{U_i \to U\}$ with each $U_i \in \mathcal{C}_H$ such that $s|_{U_i} = t|_{U_i}$. Then $s = t$ since E is separated. Hence $\epsilon(U)$ is injective. The surjectivity of $\epsilon(U)$ is proved similarly using the sheaf property of E. Suppose $\sigma = (\{U_i \to U\}, \{s_i\})$ represents an element of $(\xi_H E)(U)$. Since $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ for all i, j, there is an element $s \in E(U)$ such that $s|_{U_i} = s_i$ for all i by the sheaf property of E. Then $(\{U \xrightarrow{1} U\}, s)$ represents the same element as σ. Hence $\epsilon(U)$ is surjective. $\square$

Proposition 3.2.4. *Let X be an object of a site $\mathbb{T}$. Suppose $K \subseteq H$ are sieves on X belonging to $\mathbb{T}$, F is an H-presheaf, and $F|_K$ is the restriction of F to $\mathcal{C}_K$. Then there is a natural isomorphism $\xi_K(F|_K) \to \xi_H F$.*

Proof. We will use the notations of the proof of Theorem 3.2.2. First note that $(F|_K)^s = F^s|_K$. It is because if the pullbacks of s and t to each U_i coincide where $\mathcal{U} = \{U_i \to U\}$ is a covering that belongs to $\mathcal{C}_H$ with $U \in \mathcal{C}_K$, then there is a refinement $\{U_{ij} \to U\}$ of $\mathcal{U}$ that belongs to $\mathcal{C}_K$, so that the pullbacks of s and t to each U_{ij} coincide.

We construct a map $\xi_K(F|_K) \to \xi_H F$ as follows. Suppose U is an object of $\mathbb{T}$. An element of $\xi_K(F|_K)(U)$ is represented by a pair $\sigma = (\{U_i \to U\}, \{s_i\})$ such that each U_i is in $\mathcal{C}_K$ and $s_i \in (F|_K)^s(U_i) = F^s(U_i)$. The pair also represents an element of $\xi_H F(U)$ since $\mathcal{C}_K \subseteq \mathcal{C}_H$. Also, equivalent representatives of an element of $\xi_K(F|_K)(U)$ represent the same element of $\xi_H F(U)$. Therefore, we can define $\xi_K(F|_K)(U) \to \xi_H F(U)$ by sending $[\sigma]$ to $[\sigma]$ (same notation but classes in different equivalence relations), then it defines a map of sheaves $\xi_K(F|_K) \to \xi_H F$. From the way it is defined, we see that it is a natural map of sheaves.

Now we prove that it is an isomorphism. Suppose $\sigma = (\{U_i \to U\}, \{s_i\})$ and $\tau = (\{V_j \to V\}, \{t_j\})$ represent elements of $\xi_K(F|_K)(U)$ such that $[\sigma] = [\tau]$ in $\xi_H F(U)$. It implies that the pullbacks of s_i and t_j coincide is $U_i \times_U V_j$. But U_i, V_j, and $U_i \times_U V_j$ belong to $\mathcal{C}_K$. Therefore σ and τ represent the same element of $\xi_K(F|_K)(U)$. Hence $\xi_K F|_K(U) \to \xi_H F(U)$ is injective. To prove that it is surjective, suppose $\sigma = (\{U_i \to U\}, \{s_i\})$ represent an element of $\xi_H F(U)$. Then each U_i is in $\mathcal{C}_H$. For each U_i, there is a covering $\{U_{ij} \to U_i\}$ such that $U_{ij} \in \mathcal{C}_K$. Then $\{U_{ij} \to U\}$ is a refinement of $\{U_i \to U\}$ and the pair $\sigma' = (\{U_{ij} \to U\}, \{s_{ij}\})$ where s_{ij} is the pullback of s_i to U_{ij} represent the same element as σ does. But σ' can also represent an element of $\xi_K(F|_K)(U)$, hence $\xi_K(F|_K)(U) \to \xi_H F(U)$ is surjective. $\square$

In the big Zariski site $(Sch/X)_{Zar}$, the big structure sheaf $\mathcal{O}_X^b$ is a sheaf on $(Sch/X)_{Zar}$ such that $\mathcal{O}_X^b(Y) = \mathcal{O}_Y(Y)$ for each object $Y \xrightarrow{f} X$, and for $g : Z \to Y$ over X, $\mathcal{O}_X^b(g) : \mathcal{O}_X^b(Y) \to \mathcal{O}_X^b(Z)$ is the map of global sections $\mathcal{O}_Y(Y) \to \mathcal{O}_Z(Z)$ induced by g. We simply write $\mathcal{O}_X$ for $\mathcal{O}_X^b$. If H is a sieve on X belonging to the site, the restriction $\mathcal{O}_X|_H$ is an H-sheaf of rings.

From now on, our discussion will be specialized in Zariski topology and presheaves of modules, so an H-(pre)sheaf will mean an H-(pre)sheaf of $\mathcal{O}_X|_H$-modules unless stated otherwise. And the notations $Pre((Sch/X)_{Zar})$, $Shv((Sch/X)_{Zar})$, and so on will have the meaning of the categories of presheaves of $\mathcal{O}_X|_H$-modules, sheaves of $\mathcal{O}_X|_H$-modules, and so on.

Consider the big Zariski site $(Sch/X)_{Zar}$ on a scheme X. Suppose H is a sieve on X belonging to $(Sch/X)_{Zar}$ and F is an H-presheaf. For each object $Y \xrightarrow{f} X$ of $\mathcal{C}_H$, we can define a presheaf $F|_Y$ on the small Zariski site on Y, that is, a presheaf on Y in the usual sense. Define $F|_Y$ to be the restriction of F to Y_{zar}. In other words, $F|_Y(U_{[g]}) = F(U_{[fg]})$ for each open immersion $g : U \to Y$, and $F|_Y(h) = F(h)$ for each map $h : V \to U$ of Y_{zar}, which may be considered as a map of $\mathcal{C}_H$. We will call $F|_Y$ the restriction of F to Y along f. If G is another H-presheaf and there is a map of H-presheaves $F \to G$, we get a natural map $F|_Y \to G|_Y$. So the restriction is a functor $Pre_H((Sch/X)_{Zar}) \to Pre(Y_{zar})$.

Proposition 3.2.5. *Suppose X is a scheme, H is a sieve on X belonging to $(Sch/X)_{Zar}$, and F is an H-presheaf. If Y is an object of $\mathcal{C}_H$, then $(\xi_H F)|_Y = \xi(F|_Y)$ where ξ_H is the sheafification of H-presheaves and ξ is the sheafification of $\mathcal{O}_Y$-modules.*

Proof. By the definitions of ξ_H and ξ from Theorem 3.2.2, for each open immersion $U \to Y$, $(\xi_H F)|_Y(U) = (\xi_H F)(U)$ is the set of pairs $(\{U_i \to U\}, \{s_i\})$ modulo an equivalence relation where $\{U_i \to U\}$ is a Zariski cover, and $s_i \in F(U_i)$ for each i. Similarly, $\xi(F|_Y)$ is the set of such pairs with $s_i \in F|_Y(U_i) = F(U_i)$ modulo an equivalence relation. Both of them have the same collection of Zariski covers, and the same equivalence relations. Therefore, $(\xi_H F)|_Y(U) = \xi(F|_Y)(U)$.

The following diagram is commutative for any $V \to U$ by definition.

$$\begin{tikzcd}
(\xi_H F)|_Y(U) \arrow[r] \arrow[d, equal] & (\xi_H F)|_Y(V) \arrow[d, equal] \\
\xi(F|_Y)(U) \arrow[r] & \xi(F|_Y)(V)
\end{tikzcd}$$

This completes the proof. ☐

Now consider $(Sch/X)_{Zar}$, and let H be a sieve on X, F an H-sheaf, and $Y \in \mathcal{C}_H$. Then $F|_Y$ is a sheaf of $\mathcal{O}_Y$-modules. If $g : Z \to Y$ is a map in $\mathcal{C}_H$, then for every open immersion $U \to Y$, there is a map

$$F|_Y(U) = F(U) \xrightarrow{F(\pi_U)} F(U \times_Y Z) = F|_Z(U \times_Y Z) = g_* F|_Z(U),$$

and the diagram below induced by a map $V \to U$ commutes.

$$\begin{tikzcd}
F|_Y(U) \arrow[r] \arrow[d] & g_* F|_Z(U) \arrow[d] \\
F|_Y(V) \arrow[r] & g_* F|_Z(V)
\end{tikzcd}$$

Hence there is a map $\rho_{F,g} : F|_Y \to g_* F|_Z$. By adjointness, we get a natural map $\lambda_{F,g} : g^*(F|_Y) \to F|_Z$ of sheaves of $\mathcal{O}_Z$-modules.

We can define the extension of a sheaf from the small to the big Zariski site. Given a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, define $B\mathcal{F}$, a sheaf on $(Sch/X)_{Zar}$ by setting $B\mathcal{F}(Y) = f^*\mathcal{F}(Y)$ for each object $Y \xrightarrow{f} X$ of $(Sch/X)_{Zar}$. If $g : Z \to Y$ is a map over X, $B\mathcal{F}(g)$ is defined to be the composite

$$f^*\mathcal{F}(Y) \to g^* f^* \mathcal{F}(Z) \xrightarrow{\cong} (fg)^* \mathcal{F}(Z)$$

induced by the map of global sections. The commutativity of the following diagram shows $B\mathcal{F}(gh) = B\mathcal{F}(h) B\mathcal{F}(g)$ for $W \xrightarrow{h} Z \xrightarrow{g} Y$.

$$\begin{tikzcd}
f^*\mathcal{F}(Y) \arrow[r] \arrow[d] & (gh)^* f^* \mathcal{F}(W) \arrow[d] \arrow[rdd] & \\
g^* f^* \mathcal{F}(Z) \arrow[r] \arrow[d] & h^* g^* f^* \mathcal{F}(W) \arrow[d] & \\
(fg)^* \mathcal{F}(Z) \arrow[r] & h^*(fg)^* \mathcal{F}(W) \arrow[r] & (fgh)^* \mathcal{F}(W)
\end{tikzcd}$$

Lemma 3.2.6. *Suppose $\mathcal{F}$ is a sheaf on X_{zar}, and $B\mathcal{F}$ the extension to $(Sch/X)_{Zar}$. Then*

(1) for each object $Y \xrightarrow{f} X$ of $(Sch/X)_{Zar}$, there is a natural isomorphism $B\mathcal{F}|_Y \to f^*\mathcal{F}$, (in particular, $B\mathcal{F}$ is a sheaf,)

(2) for each map $g : Z \to Y$ over X, the induced map $g^*(B\mathcal{F}|_Y) \to B\mathcal{F}|_Z$ is an isomorphism.

Proof. Suppose $g : U \to Y$ is an open immersion. Then $B\mathcal{F}|_Y(U) = B\mathcal{F}(U) = (fg)^*\mathcal{F}(U)$. Define $B\mathcal{F}|_Y(U) \to f^*Y(U)$ to be the composite $(fg)^*\mathcal{F}(U) \xrightarrow{\cong} g^* f^* \mathcal{F}(U) \xrightarrow{\cong} f^*\mathcal{F}(U)$. If $h : V \to U$ is a map such that gh is an open immersion, then the diagram

$$\begin{tikzcd}
(fg)^*\mathcal{F}(U) \arrow[r] \arrow[d] & g^*f^*\mathcal{F}(U) \arrow[r] \arrow[d] & f^*\mathcal{F}(U) \arrow[dd] \\
h^*(fg)^*\mathcal{F}(V) \arrow[r] \arrow[d] & h^*g^*f^*\mathcal{F}(V) \arrow[d] & \\
(fgh)^*\mathcal{F}(V) \arrow[r] & (gh)^*f^*\mathcal{F}(V) \arrow[r] & f^*\mathcal{F}(V)
\end{tikzcd}$$

commutes. All of the maps involved in the diagram are natural in $\mathcal{F}$. This proves the first statement. For the second, note that the following diagram commutes.

$$\begin{tikzcd}
g^*(B\mathcal{F}|_Y) \arrow[r] \arrow[d, "\cong"'] & B\mathcal{F}|_Z \arrow[d, "\cong"] \\
g^*f^*\mathcal{F} \arrow[r, "\cong"'] & (fg)^*\mathcal{F}
\end{tikzcd}$$

Three isomorphisms in the diagram implies that the top arrow is an isomorphism. $\square$

Since the definition of B is functorial in $\mathcal{F}$, we have defined a functor

$$B : Shv(X_{zar}) \to Shv((Sch/X)_{Zar}).$$

Lemma 3.2.7. *Suppose F is a sheaf on $(Sch/X)_{Zar}$ such that the induced map $\lambda_{F,f} : f^*(F|_X) \to F|_Y$ is an isomorphism for every object $Y \xrightarrow{f} X$ of $(Sch/X)_{Zar}$. Then there is an isomorphism $\eta : B(F|_X) \to F$ that is natural in the sense that if G is another such sheaf, and there is a map $\alpha : F \to G$, then the induced diagram commutes.*

$$\begin{tikzcd}
B(F|_X) \arrow[r, "\eta"] \arrow[d, "B\alpha|_X"'] & F \arrow[d, "\alpha"] \\
B(G|_X) \arrow[r, "\eta"'] & G
\end{tikzcd}$$

Proof. For each object $Y \xrightarrow{f} X$, define $\eta(Y) = \lambda_{F,f}(Y)$.

$$B(F|_X)(Y) = f^*(F|_X)(Y) \xrightarrow[\cong]{\lambda_{F,f}(Y)} F|_Y(Y) = F(Y)$$

To show that η is an isomorphism of functors, we need to show the commutativity of the following diagram for every map $g : Z \to Y$ over X.

$$\begin{tikzcd} B(F|_X)(Y) \arrow[r] \arrow[d] & F(Y) \arrow[d] \\ B(F|_X)(Z) \arrow[r] & F(Z) \end{tikzcd}$$

It is enough to show the commutativity of the following diagram.

$$\begin{tikzcd} f^*(F|_X)(Y) \arrow[r, "\lambda_f(Y)"] \arrow[d] & F|_Y(Y) \arrow[d] \arrow[dd, bend left, "F(g)"] \\ g^*f^*(F|_X)(Z) \arrow[r, "g^*\lambda_f(Z)"] \arrow[d] & g^*(F|_Y)(Z) \arrow[d, "\lambda_g(Z)"'] \\ (fg)^*(F|_X)(Z) \arrow[r, "\lambda_{fg}(Z)"'] & F|_Z(Z) \end{tikzcd}$$

The top square is commutative since it is induced by the natural transformation $1 \to g_* g^*$. The bottom square is commutative since the corresponding diagram of sheaves before taking the global sections commutes. On the level of stalks, it corresponds to the diagram of modules

$$\begin{tikzcd} C \otimes_B (B \otimes_A M) \arrow[r] \arrow[d] & C \otimes_A N \arrow[d] \\ C \otimes_A M \arrow[r] & L \end{tikzcd}$$

induced by rings A, B, C, an A-module M, a B-module N, and a C-module L, together with morphisms of rings $A \to B \to C$, a B-linear map $M \to N$, and a C-linear map $N \to L$. Finally, the part on the right is obtained by taking the global sections of the following diagram of sheaves,

$$\begin{tikzcd} F|_Y \arrow[d] \arrow[dr, "\rho_g"] & \\ g_*g^*F|_Y \arrow[r, "g_*\lambda_g"'] & g_*F|_Z \end{tikzcd}$$

which is commutative since ρ and λ corresponds to each other in the adjoint relationship of g_* and g^*. The naturality of η follows from the naturality of $\lambda_{F,f}$. $\square$

Let $f : Y \to X$ be a map of schemes and H a sieve on X belonging to $(Sch/X)_{Zar}$. We will define the pullback functor $f^* : Pre_H((Sch/X)_{Zar}) \to Pre_{f^*H}((Sch/Y)_{Zar})$. Recall that f^*H is the sieve on Y belonging to $(Sch/Y)_{Zar}$ such that $Z \xrightarrow{g} Y$ is an object of $\mathcal{C}_{f^*H}$ if and only if the composition

$Z \xrightarrow{g} Y \xrightarrow{f} X$ is in $\mathcal{C}_H$. Therefore, there is a functor $f_* : \mathcal{C}_{f^*H} \to \mathcal{C}_H$ defined by composition with f. Then for an H-presheaf E, f^*E is defined to be Ef_*^{op}, that is, $f^*E(Z_{[g]}) = E(Z_{[fg]})$ for each object $Z \xrightarrow{g} Y$ of $\mathcal{C}_{f^*H}$ and $f^*E(h) = E(h)$ for each morphism h of $\mathcal{C}_{f^*H}$, which may be considered as a morphism of $\mathcal{C}_H$ as well. In addition, for a map $\alpha : E \to F$ of H-presheaves, $f^*\alpha : f^*E \to f^*F$ is defined by
$(f^*\alpha)(Z_{[g]}) = \alpha(Z_{[fg]})$ for each object $Z \xrightarrow{g} Y$ of $\mathcal{C}_{f^*H}$. The following diagram commutes for any morphism h of $\mathcal{C}_H$, thus $f^*\alpha$ is indeed a map of f^*H-presheaves.

$$\begin{tikzcd}
f^*E(Z) \arrow[r, equal] \arrow[d, "f^*E(h)"'] & E(Z) \arrow[r, "\alpha(Z)"] \arrow[d, "E(h)"'] & F(Z) \arrow[r, equal] \arrow[d, "F(h)"] & f^*F(Z) \arrow[d, "f^*F(h)"] \\
f^*E(W) \arrow[r, equal] & E(W) \arrow[r, "\alpha(W)"'] & F(W) \arrow[r, equal] & f^*F(W)
\end{tikzcd}$$

If $\beta : F \to G$ is another map of H-presheaves, then $f^*(\alpha\beta) = f^*\alpha f^*\beta$ as can be seen easily by definition. Therefore, f^* is a functor. Now a series of lemmas investigating the properties of the functor f^* follows.

Lemma 3.2.8. *Let H be a sieve on X belonging to $(Sch/X)_{Zar}$, E an H-presheaf, and $Y \xrightarrow{f} X$ a map of schemes. Then $(f^*E)|_Z = E|_Z$ for any object $Z \xrightarrow{g} Y$ of $\mathcal{C}_{f^*H}$.*

Proof. This follows directly from the definitions of the pullback functor and the restriction functor. □

Lemma 3.2.9. *Let $f : Y \to X$ and $g : Z \to Y$ be maps of schemes and H a sieve on X belonging to $(Sch/X)_{Zar}$. Then $(fg)^*H$ and g^*f^*H are the same sieves, and $(fg)^* = g^*f^*$ (equality, not natural isomorphism) as functors from $Pre_H((Sch/X)_{Zar})$ to $Pre_{(fg)^*H}((Sch/Z)_{Zar})$.*

Proof. A map $h : W \to Z$ is in the sieve $(fg)^*H$ if and only if fgh is in H, and this condition is equivalent for h to be in the sieve g^*f^*H. Hence $(fg)^*H = g^*f^*H$. The functors $(fg)_*$ and f_*g_* from $\mathcal{C}_{(fg)^*H}$ to $\mathcal{C}_H$ are the same because both are defined by composition with fg. Therefore, for every H-presheaf E,

$$(fg)^*E = E(fg)_*^{op} = Ef_*^{op}g_*^{op} = g^*f^*E,$$

and for every morphism $\alpha : E \to F$ and every object $W \xrightarrow{h} Z$ of $\mathcal{C}_{(fg)^*H}$,

$$((fg)^*\alpha)(W_{[h]}) = \alpha(W_{[fgh]}) = (f^*\alpha)(W_{[gh]}) = g^*f^*\alpha(W_{[h]}).$$

□

Lemma 3.2.10. *Suppose $f : Y \to X$ is a map of schemes and $\mathcal{F}$ an $\mathcal{O}_X$-module. Then there is a natural* *isomorphism* $Bf^*\mathcal{F} \xrightarrow{\cong} f^*B\mathcal{F}$ *where the first* B *is the extension functor* $Pre(Y_{zar}) \to Pre((Sch/Y)_{Zar})$ *and the second is* $Pre(X_{zar}) \to Pre((Sch/X)_{Zar})$.

Proof. For each scheme $Z \xrightarrow{g} Y$ over Y, $(Bf^*\mathcal{F})(Z_{[g]}) = (g^*f^*\mathcal{F})(Z)$, and $(f^*B\mathcal{F})(Z_{[g]}) = (B\mathcal{F})(Z_{[fg]}) = ((fg)^*\mathcal{F})(Z)$. Define $\alpha_g : (Bf^*\mathcal{F})(Z_{[g]}) \to (f^*B\mathcal{F})(Z_{[g]})$ to be the map $(g^*f^*\mathcal{F})(Z) \to ((fg)^*\mathcal{F})(Z)$ induced by the natural isomorphism $g^*f^* \to (fg)^*$. If $h : W \to Z$ is any map over Y, then the following diagram commutes.

$$\begin{tikzcd}
(g^*f^*\mathcal{F})(Z) \arrow[r, "\alpha_g", "\cong"'] \arrow[d] \arrow[dd, bend right=60, "(Bf^*\mathcal{F})(h)"'] & ((fg)^*\mathcal{F})(Z) \arrow[d] \arrow[dd, bend left=60, "(f^*B\mathcal{F})(h)"] \\
(h^*g^*f^*\mathcal{F})(W) \arrow[r, "\cong"] \arrow[d, "\cong"] & (h^*(fg)^*\mathcal{F})(W) \arrow[d, "\cong"'] \\
((gh)^*f^*\mathcal{F})(W) \arrow[r, "\alpha_{gh}", "\cong"'] & ((fgh)^*\mathcal{F})(W)
\end{tikzcd}$$

□

Lemma 3.2.11. *Let* $f : Y \to X$ *be a map of schemes,* H *a sieve on* X *belonging to* $(Sch/X)_{Zar}$*. Then* $\xi_{f^*H}f^* = f^*\xi_H$ *as functors from* $Pre_H((Sch/X)_{Zar})$ *to* $Pre((Sch/Y)_{Zar})$.

$$\begin{array}{ccccc}
Pre_H((Sch/X)_{Zar}) & \xrightarrow{f^*} & Pre_{f^*H}((Sch/Y)_{Zar}) & \xrightarrow{\xi_{f^*H}} & Pre((Sch/Y)_{Zar}) \\
Pre_H((Sch/X)_{Zar}) & \xrightarrow{\xi_H} & Pre((Sch/X)_{Zar}) & \xrightarrow{f^*} & Pre((Sch/Y)_{Zar})
\end{array}$$

Proof. Suppose E is an H-presheaf and $U \xrightarrow{g} Y$ an object of $(Sch/Y)_{Zar}$. By the definition of the sheafification functor in Theorem 3.2.2, $(\xi_{f^*H}f^*E)(U)$ is the set of equivalence classes $[\{U_i \to U\}, \{s_i\}]$ such that for each i, $U_i \to U \xrightarrow{g} Y$ is an object of f^*H, $s_i \in (f^*E)^s(U_i)$, and $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ for every i, j. Two pairs $(\{U_i \to U\}, \{s_i\})$ and $(\{V_j \to U\}, \{t_j\})$ represent the same class if and only if the pullbacks of s_i and t_j to $U_i \times_U V_j$ coincide. Now note that $U_i \to U \xrightarrow{g} Y$ is an object of f^*H if and only if $U_i \to U \xrightarrow{fg} X$ is an object of H, and $(f^*E)^s(U_i) = E^s(U_i)$. Therefore, a pair $(\{U_i \to U\}, \{s_i\})$ represents an element of $(\xi_{f^*H}f^*E)(U)$ if and only if it represents an element of $(\xi_H E)(U) = (f^*\xi_H E)(U)$. Also, the equivalence relations defining $(\xi_{f^*H}f^*E)(U)$ and $(f^*\xi_H E)(U)$ are the same. Therefore, $(\xi_{f^*H}f^*E)(U) = (f^*\xi_H E)(U)$. Next, if $h : V \to U$ is a morphisms of $(Sch/Y)_{Zar}$, then both $(\xi_{f^*H}f^*E)(h)$ and $(f^*\xi_H E)(h)$ send the class $[\{U_i \to U\}, \{s_i\}]$ to the class $[\{U_i \times_U V\}, \{s_i|_{U_i \times_U V}\}]$. Therefore, $(\xi_{f^*H}f^*E)(h) = (f^*\xi_H E)(h)$. This shows that $\xi_{f^*H}f^*$ and $f^*\xi_H$ agree on objects. To show that they also agree on morphisms, suppose $\alpha : E \to F$ is a map of H-presheaves. For each object $U \xrightarrow{g} Y$ of $(Sch/Y)_{Zar}$, both $(\xi_{f^*H}f^*\alpha)(U)$ and $(f^*\xi_H\alpha)(U)$ send the class $[\{U_i \to U\}, \{s_i\}]$ to $[\{U_i \to U\}, \{\alpha^s s_i\}]$. Therefore, $\xi_{f^*H}f^*\alpha = f^*\xi_H\alpha$. This completes the proof. $\square$

3.3 Standard vector bundles

In this section, we give the definition of the category of standard vector bundles and prove its properties. This category is equivalent to the category of usual vector bundles and satisfies various strict functoriality. Among other things, it has strictly functorial pullback functor and strictly associative tensor product, which is also strictly commutative with line bundles. The construction of such a category was built upon the the notion of big vector bundles [6, C.4], [9, p.169], and is an original work of the author. All sheaves are assumed to be sheaves of modules.

3.3.1 The definition of standard vector bundles

When A is a commutative ring, we will call an A-module of the form

$$A^n = \{(a_1, \ldots, a_n) | a_i \in A, i = 1, \ldots, n\}$$

a *standard free* A-module. A finitely generated A-module is free if and only if it is isomorphic to a standard free module. For a scheme Y and a presheaf E on the big Zariski site $(Sch/Y)_{Zar}$, a map $\mathcal{O}_Y^n \to E$ is completely determined by n elements of $E(Y)$, the images of the standard basis vectors of $\mathcal{O}_Y(Y)^n$. Suppose H is a sieve on a scheme X belonging to $(Sch/X)_{Zar}$, and suppose E is an H-presheaf. If $Y \xrightarrow{f} X$ is an object of $\mathcal{C}_H$, then $f^*H = H_Y$ where H_Y is the sieve $\mathrm{Hom}(-, Y)$, and f^*E is a presheaf on the big Zariski site $(Sch/Y)_{Zar}$. If, furthermore, $E(Y)$ is a standard free module, then the standard basis of $E(Y) = f^*E(Y)$ induces a map $\mathcal{O}_Y^n \to f^*E$ of presheaves on $(Sch/Y)_{Zar}$ such that the map on Y is the identity.

Definition 3.3.1. Suppose X is a scheme. A *standard vector bundle on* X is a pair (H, E) where H is a sieve on X that belongs to the site $(Sch/X)_{Zar}$, and E is an H-presheaf on $(Sch/X)_{Zar}$ satisfying the following property: for each object $Y \xrightarrow{f} X$ of $\mathcal{C}_H$, there exists an integer n such that $E(Y) = \mathcal{O}_Y(Y)^n$, i.e., $E(Y)$ is a standard free module, and the map $\epsilon_f : \mathcal{O}_Y^n \to f^*E$ induced by the standard basis of $E(Y)$ is an isomorphism. If the integer n is the same for all objects of $\mathcal{C}_H$, then it is called the *rank* of E. A standard vector bundle of rank 1 is called a *standard line bundle*. The category of standard vector bundles on X is denoted by $\mathbf{V}(X)$. The set of morphisms from (H, E) to (K, F) is defined to be the set of morphisms between the associated sheaves,

$$\mathrm{Hom}_{\mathbf{V}(X)}((H,E),(K,F)) = \mathrm{Hom}_{Shv((Sch/X)_{Zar})}(\xi_H E, \xi_K F).$$

In this definition, we required the value of E at every object to be a standard free module. This is the key requirement for the properties listed in Theorem 3.3.8. For simpler notation, we will sometimes write E for (H,E). When we do so, we will call E a standard vector bundle and H the associated sieve, or we will simply call E an *H-vector bundle* (or *H-line bundle* if the rank is 1). Note that E is actually an H-sheaf, not just an H-presheaf since every pullback of E is a sheaf. If $g : Z \to Y$ is a morphism of $\mathcal{C}_H$, then $E(Y)$ and $E(Z)$ have the same rank.

Example 3.3.2. The simplest example of standard vector bundles is the *trivial* standard vector bundle $\mathcal{O}_X^n$ of rank $n \geq 0$. It is defined as an H_X-vector bundle. For each object $Y \to X$ of $(Sch/X)_{Zar}$, $\mathcal{O}_X^n(Y) = \mathcal{O}_Y(Y)^n$, and for each map $g : Z \to Y$ over X, the restriction map $\mathcal{O}_X^n(g) : \mathcal{O}_X^n(Y) \to \mathcal{O}_X^n(Z)$ is induced by the map $\mathcal{O}_Y(Y) \to \mathcal{O}_Z(Z)$ of global sections of the structure sheaves. The trivial standard vector bundle of rank 0 will be denoted by 0 and called the zero bundle.

There is a way to produce a standard vector bundle from a locally free sheaf on a scheme. The next lemma is useful for various constructions in this section. It says that a *locally free* H-presheaf can be standardized by choosing trivialization data.

Lemma 3.3.3. *Let X be a scheme, H a sieve on X belonging to $(Sch/X)_{Zar}$, and E an H-presheaf. Suppose there is an integer n_f and an isomorphism $\varphi_f : \mathcal{O}_Y^{n_f} \to f^*E$ for each object $Y \xrightarrow{f} X$ of $\mathcal{C}_H$. Then there exists an H-vector bundle $S_H^\varphi E$, and an isomorphism $\varphi_E : S_H^\varphi E \cong E$ induced by φ.*

Proof. For an object $Y \xrightarrow{f} X$ of $\mathcal{C}_H$, define $S_H^\varphi E(Y) = \mathcal{O}_Y(Y)^{n_f}$. For a morphism $g : Z \to Y$ of $\mathcal{C}_H$, define $S_H^\varphi E(g)$ to be the composite map

$$\mathcal{O}_Y(Y)^{n_f} \xrightarrow[\cong]{\varphi_f(Y)} (f^*E)(Y) = E(Y) \xrightarrow{E(g)} E(Z) = ((fg)^*E)(Z) \xrightarrow[\cong]{\varphi_{fg}^{-1}(Z)} \mathcal{O}_Z(Z)^{n_{fg}}.$$

If h is another morphism of $\mathcal{C}_H$, then $S_H^\varphi E(gh) = S_H^\varphi E(h) S_H^\varphi E(g)$. Hence $S_H^\varphi E$ is an H-presheaf. From the way $S_H^\varphi E$ is defined on morphisms, we see that the map $\varphi_E : S_H^\varphi E \to E$ defined by $\varphi_E(Y) : S_H^\varphi E(Y) \xrightarrow{\varphi_f(Y)} E(Y)$ on each object $Y \xrightarrow{f} X$ of $\mathcal{C}_H$ is an isomorphism of H-presheaves. Let $\epsilon : \mathcal{O}_Y^{n_f} \to f^* S_H^\varphi E$ be the map induced by the standard basis of $S_H^\varphi E(Y)$. Then the following diagram commutes

$$\begin{tikzcd}
\mathcal{O}_Y^{n_f} \arrow[r, "\epsilon"] \arrow[dr, "\varphi_f"'] & f^*S_H^{\varphi}E \arrow[d, "f^*\varphi_E"] \\
& f^*E
\end{tikzcd}$$

because the diagram of global sections commute.

$$\begin{tikzcd}
\mathcal{O}_Y(Y)^{n_f} \arrow[r, "1"] \arrow[dr, "\varphi_f(Y)"'] & \mathcal{O}_Y(Y)^{n_f} \arrow[d, "\varphi_f(Y)"] \\
& E(Y)
\end{tikzcd}$$

Since φ_f and $f^*\varphi_E$ are isomorphisms, so is ϵ. $\square$

Let X be a scheme and $\mathcal{E}$ a locally free sheaf of finite rank on X_{zar}. We can construct a standard vector bundle from $\mathcal{E}$ once we make certain choices. Suppose that $\mathcal{U} = \{U_i \to U\}$ is a covering such that $\mathcal{E}|_{U_i}$ is a free $\mathcal{O}_{U_i}$-module for each i. Let H be the sieve associated to $\mathcal{U}$. If $Y \xrightarrow{f} X$ is an object of $\mathcal{C}_H$, then f factors as $Y \to U_i \to X$ for some i. Hence $f^*\mathcal{E}$ is a free $\mathcal{O}_Y$-module of finite rank. We choose an isomorphism $\alpha_f : \mathcal{O}_Y^n \to f^*\mathcal{E}$ for every object $Y \xrightarrow{f} X$ of $\mathcal{C}_H$. Since $f^*H = H_Y = f^*H_X$, $f^*(B\mathcal{E}|_H) = f^*B\mathcal{E}$, and by Lemma 3.2.10, $f^*B\mathcal{E} \cong Bf^*\mathcal{E}$. Then define φ_f to be the composite map

$$\mathcal{O}_Y^n \cong B\mathcal{O}_Y^n \xrightarrow[\cong]{B\alpha_f} Bf^*\mathcal{E} \xrightarrow[\cong]{} f^*(B\mathcal{E}|_H).$$

Corollary 3.3.4. *Let X be a scheme and $\mathcal{E}$ a locally free sheaf of finite rank on X_{zar}. If we choose H and φ as described in the previous paragraph, then $S_H^{\varphi}B\mathcal{E}|_H$ is a standard vector bundle. Moreover, there is an isomorphism $\gamma_{\mathcal{E}} : \xi_H S_H^{\varphi}B\mathcal{E}|_H \to B\mathcal{E}$ of sheaves on $(Sch/X)_{Zar}$.*

Proof. Applying Lemma 3.3.3 to $E = B\mathcal{E}|_H$, we get a standard vector bundle $S_H^{\varphi}B\mathcal{E}|_H$, and an isomorphism $\varphi_{B\mathcal{E}|_H} : S_H^{\varphi}B\mathcal{E}|_H \to B\mathcal{E}|_H$. The isomorphism $\gamma_{\mathcal{E}}$ is the composite map

$$\xi_H S_H^{\varphi}B\mathcal{E}|_H \xrightarrow[\cong]{\xi_H\varphi} \xi_H(B\mathcal{E}|_H) \xrightarrow{\cong} \xi_{H_X}B\mathcal{E} \xrightarrow{\cong} B\mathcal{E}$$

where the second and the third isomorphisms are from Proposition 3.2.4 and the fact that $B\mathcal{E}$ is a sheaf. $\square$

Now we want to define a pullback functor $\mathbf{V}(X) \to \mathbf{V}(Y)$ induced by a map $f : Y \to X$ of schemes. If E is an H-vector bundle, then the f^*H-presheaf f^*E is an f^*H-vector bundle. To prove this, suppose

$Z \xrightarrow{g} Y$ is an object of f^*H. We need to prove that $f^*E(Z_{[g]})$ is a standard free module and that the map ${\mathcal{O}_Z}^n \to g^*f^*E$ induced by the standard basis of $f^*E(Z_{[g]})$ is an isomorphism. But those follow from the condition of E being an H-vector bundle since $Z \xrightarrow{g} Y \xrightarrow{f} X$ is an object of H, $f^*E(Z_{[g]}) = E(Z_{[fg]})$, and $g^*f^*E = (fg)^*E$ by Lemma 3.2.9. Therefore we can define the pullback of (H, E) to be (f^*H, f^*E). Suppose $\alpha : (H, E) \to (K, F)$ is a morphism of standard vector bundles in $\mathbf{V}(X)$, that is, a morphism $\alpha : \xi_H E \to \xi_K F$ of sheaves. Then $f^*\alpha$ is a morphism $f^*\xi_H E \to f^*\xi_K F$, which is a morphism $\xi_{f^*H} f^*E \to \xi_{f^*K} f^*F$ by Lemma 3.2.11. So it is a map $(f^*H, f^*E) \to (f^*K, f^*F)$ of standard vector bundles. The pullback of the map α of standard vector bundles is defined to be $f^*\alpha$. If $\beta : (K, F) \to (L, G)$ is another map of standard vector bundles, then $f^*(\beta\alpha) = f^*\beta f^*\alpha$ since f^* is a functor. Therefore, we have defined a functor $\mathbf{V}(X) \to \mathbf{V}(Y)$. We will denote it by f^*.

Proposition 3.3.5. *Suppose $f : Y \to X$ and $g : Z \to Y$ are maps of schemes. Then $(fg)^* = f^*g^*$ as functors $\mathbf{V}(Z) \to \mathbf{V}(X)$. (This is a strict equality, not a natural isomorphism.)*

Proof. This follows from the definition of the pullback functors and Lemma 3.2.9. □

Lemma 3.3.6. *Suppose (H, E) is a standard vector bundle on X. If $f : Y \to X$ is a map of schemes, then the induced map $\lambda : f^*(\xi_H E|_X) \to \xi_H E|_Y$ is an isomorphism.*

Proof. We prove it by showing that the map at every stalk is an isomorphism. Suppose $y \in Y$ and $x = f(y)$. We can choose an open subscheme $U \subset X$ containing x such that the inclusion $i : U \to X$ is in the sieve H. Let V be an open subscheme of Y containing $f^{-1}(U)$, and $j : V \to Y$ the inclusion, and $g = f|_V : V \to U$.

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ U \arrow[r, "i"'] & X \end{tikzcd}$$

Since E is an H-vector bundle, there is an isomorphism $\epsilon : \mathcal{O}_U^n \to i^*E$ induced by the standard basis of $E(U)$. Since $i^*E|_U = E|_U$ and $i^*E|_V = E|_V$ (Lemma 3.2.8), we obtain the following commutative diagram, which shows that the induced map $\lambda_E : g^*E|_U \to E|_V$ is an isomorphism.

$$\begin{tikzcd} \mathcal{O}_V^n \arrow[r, "\cong"] \arrow[dr, "1"'] & g^*\mathcal{O}_U^n \arrow[r, "g^*\epsilon|_U", "\cong"'] \arrow[d, "\lambda_{\mathcal{O}}"] & g^*E|_U \arrow[d, "\lambda_E"] \\ & \mathcal{O}_V^n \arrow[r, "\cong", "\epsilon|_V"'] & E|_V \end{tikzcd}$$

Now $j^*f^*(\xi_H E|_X) \cong g^*i^*(\xi_H E|_X) \cong g^*(\xi_H E|_U) \cong g^*E|_U$ and $j^*(\xi_H E|_Y) \cong \xi_H E|_V \cong E|_V$. Thus we have the following commutative diagram, which shows that $\lambda|_V$ is an isomorphism.

$$\begin{tikzcd}
f^*(\xi_H E|_X)|_V \arrow[r, "\lambda|_V"] \arrow[d, "\cong"'] & \xi_H E|_V \arrow[d, "\cong"] \\
g^* E|_U \arrow[r, "\cong", "\lambda_E"'] & E|_V
\end{tikzcd}$$

Therefore, the localized map λ_y is an isomorphism as it is the localization of the top row at y. $\square$

3.3.2 Direct sum and tensor product

We will define two bifunctors $\oplus : \mathbf{V}(X) \times \mathbf{V}(X) \to \mathbf{V}(X)$ and $\otimes : \mathbf{V}(X) \times \mathbf{V}(X) \to \mathbf{V}(X)$ called direct sum and tensor product. First we define presheaf versions of direct sum and tensor product.

$$\widetilde{\oplus} : Pre_H((Sch/X)_{Zar}) \times Pre_H((Sch/X)_{Zar}) \to Pre_H((Sch/X)_{Zar})$$
$$\widetilde{\otimes} : Pre_H((Sch/X)_{Zar}) \times Pre_H((Sch/X)_{Zar}) \to Pre_H((Sch/X)_{Zar})$$

If E and F are H-presheaves, then $(E \,\widetilde{\oplus}\, F)(Y) = E(Y) \oplus F(Y)$ for each object Y, and $(E \,\widetilde{\oplus}\, F)(g) = E(g) \oplus F(g)$ for each morphism g. If $\gamma : E \to E'$ and $\delta : F \to F'$ are maps of presheaves, then

$$(\gamma \,\widetilde{\oplus}\, \delta)(Y) = \gamma(Y) \oplus \delta(Y) : E(Y) \oplus F(Y) \to E'(Y) \oplus F'(Y).$$

Similarly, $(E \,\widetilde{\otimes}\, F)(Y) = E(Y) \otimes_{\mathcal{O}_Y(Y)} F(Y)$ on objects, $(E \,\widetilde{\otimes}\, F)(g) = E(g) \otimes F(g)$ on morphisms, and

$$(\gamma \,\widetilde{\otimes}\, \delta)(Y) = \gamma(Y) \otimes \delta(Y) : E(Y) \otimes_{\mathcal{O}_Y(Y)} F(Y) \to E'(Y) \otimes_{\mathcal{O}_Y(Y)} F'(Y).$$

Let H and K be sieves on X that belong to $(Sch/X)_{Zar}$, and let E be an H-vector bundle and F a K-vector bundle. We will define their direct sum $E \oplus F$ as an $H \cap K$-vector bundle. The presheaf direct sum $E \,\widetilde{\oplus}\, F$ is not a standard vector bundle since the value at an object is not a standard free module. But we can make it into one through a standardization process (Lemma 3.3.3). For each object $Y \xrightarrow{f} X$ of $\mathcal{C}_{H \cap K}$, we have isomorphisms $\alpha : \mathcal{O}_Y^r \to f^*E$ and $\beta : \mathcal{O}_Y^s \to f^*F$ induced by the standard bases of $E(Y)$ and $F(Y)$. Let φ_f be the composite map

$$\varphi_f : \mathcal{O}_Y^{r+s} \xrightarrow[\cong]{\sigma} \mathcal{O}_Y^r \,\widetilde{\oplus}\, \mathcal{O}_Y^s \xrightarrow[\cong]{\alpha \,\widetilde{\oplus}\, \beta} f^*E \,\widetilde{\oplus}\, f^*F = f^*(E \,\widetilde{\oplus}\, F)$$

where σ is the isomorphism

$$(a_1, \ldots, a_r, a_{r+1}, \ldots, a_{r+s}) \mapsto ((a_1, \ldots, a_r), (a_{r+1}, \ldots, a_{r+s})). \tag{3.1}$$

Then define $E \oplus F = S^{\varphi}_{H\cap K}(E \mathbin{\widetilde{\oplus}} F)$. Since $E \oplus F \cong E \mathbin{\widetilde{\oplus}} F$, there is an isomorphism

$$\omega : \xi_{H\cap K}(E \oplus F) \cong \xi_{H\cap K}(E \mathbin{\widetilde{\oplus}} F) \cong \xi_H E \mathbin{\widetilde{\oplus}} \xi_K F.$$

If $\gamma : (H, E) \to (H'E')$ and $\delta : (K, F) \to (K', F')$ are maps of standard vector bundles, that is, maps $\gamma : \xi_H E \to \xi_{H'} E'$ and $\delta : \xi_K F \to \xi_{K'} F'$ of associated sheaves, then $\gamma \oplus \delta$ is defined to be the following composite map.

$$\xi_{H\cap K}(E \oplus F) \xrightarrow{\omega} \xi_H E \mathbin{\widetilde{\oplus}} \xi_K F \xrightarrow{\gamma \mathbin{\widetilde{\oplus}} \delta} \xi_{H'} E' \mathbin{\widetilde{\oplus}} \xi_{K'} F' \xrightarrow{\omega^{-1}} \xi_{H'\cap K'}(E' \oplus F')$$

If $\gamma' : (H', E') \to (H', E'')$ and $\delta' : (K', F') \to (K'', F'')$ are another pair of maps of standard vector bundles, then $(\gamma' \oplus \delta')(\gamma \oplus \delta) = \gamma'\gamma \oplus \delta'\delta$ since a similar formula for $\widetilde{\oplus}$ holds. Therefore, we have defined a bifunctor $\oplus : \mathbf{V}(X) \times \mathbf{V}(X) \to \mathbf{V}(X)$. The isomorphism ω also allows us to define projections and injections of standard vector bundles

$$\begin{aligned} p_E &: E \oplus F \to E \qquad & i_E &: E \to E \oplus F \\ p_F &: E \oplus F \to F \qquad & i_F &: F \to E \oplus F \end{aligned}$$

such that $i_E p_E + i_F p_F = 1_{E\oplus F}$, $p_E i_E = 1_E$, $p_F i_F = 1_F$, $p_E i_F = 0$, and $p_F i_E = 0$. So the direct sum operation $\oplus$ is a biproduct operation in $\mathbf{V}(X)$. This construction can be generalized to the direct sum of multiple terms. The category $\mathbf{V}(X)$ is an additive category with $\oplus$ as the biproduct operation.

The construction of the tensor product operation $\otimes : \mathbf{V}(X) \times \mathbf{V}(X) \to \mathbf{V}(X)$ is similar. If E is an H-vector bundle and F is a K-vector bundle, then $E \otimes F$ will be an $H \cap K$-vector bundle. For each object $Y \xrightarrow{f} X$ of $\mathcal{C}_{H\cap K}$, let $\alpha : \mathcal{O}^r_Y \to f^*E$ and $\beta : \mathcal{O}^s_Y \to f^*F$ be the isomorphisms induced by the standard bases of $E(Y)$ and $F(Y)$. Define φ_f to be the composite map

$$\varphi_f : \mathcal{O}^{rs}_Y \xrightarrow[\cong]{\pi^{-1}} \mathcal{O}^r_Y \mathbin{\widetilde{\otimes}} \mathcal{O}^s_Y \xrightarrow[\cong]{\alpha \mathbin{\widetilde{\otimes}} \beta} f^*E \mathbin{\widetilde{\otimes}} f^*F = f^*(E \mathbin{\widetilde{\otimes}} F)$$

where π is the isomorphism

$$(a_1, \ldots, a_r) \otimes (b_1, \ldots, b_s) \mapsto (a_1b_1, \ldots, a_1b_s, \ldots, a_rb_1, \ldots, a_rb_s). \tag{3.2}$$

Using Lemma 3.3.3 with this collection of isomorphisms, define the tensor product of E and F to be $E \otimes F = S^{\varphi}_{H\cap K}(E \mathbin{\widetilde{\otimes}} F)$. Suppose $\gamma : (H, E) \to (H'E')$ and $\delta : (K, F) \to (K', F')$ are maps of standard vector bundles. They are the maps $\gamma : \xi_H E \to \xi_{H'} E'$ and $\delta : \xi_K F \to \xi_{K'} F'$ of the associated sheaves. Since E is an H-sheaf, there is a natural isomorphism $E \cong (\xi_H E)|_H$ by Lemma 3.2.3. There are similar isomorphisms for other standard vector bundles as well. Then $E \mathbin{\widetilde{\otimes}} F \cong (\xi_H E)|_H \mathbin{\widetilde{\otimes}} (\xi_K F)|_K = \xi_H E|_{H\cap K} \mathbin{\widetilde{\otimes}} \xi_H F|_{H\cap K} = (\xi_H E \mathbin{\widetilde{\otimes}} \xi_K F)|_{H\cap K}$, and by Proposition 3.2.4, there is an isomorphism ζ defined by composing a series of isomorphisms.

$$\zeta : \xi_{H\cap K}(E \otimes F) \cong \xi_{H\cap K}(E \mathbin{\widetilde{\otimes}} F) \cong \xi_{H\cap K}((\xi_H E \mathbin{\widetilde{\otimes}} \xi_K F)|_{H\cap K}) \cong \xi_{H_X}(\xi_H E \mathbin{\widetilde{\otimes}} \xi_K F). \tag{3.3}$$

Now the map $\gamma \otimes \delta$ is defined to be the composite map

$$\begin{aligned} \gamma \otimes \delta : \xi_{H\cap K}(E \otimes F) &\xrightarrow{\zeta} \xi_{H_X}(\xi_H E \mathbin{\widetilde{\otimes}} \xi_K F) \\ &\xrightarrow[\xi(\gamma \mathbin{\widetilde{\otimes}} \delta)]{} \xi_{H_X}(\xi_{H'} E' \mathbin{\widetilde{\otimes}} \xi_{K'} F') \xrightarrow[\zeta^{-1}]{} \xi_{H'\cap K'}(E' \otimes F') \end{aligned}$$

If $\gamma' : (H', E') \to (H', E'')$ and $\delta' : (K', F') \to (K'', F'')$ are another pair of maps of standard vector bundles, then $(\gamma' \otimes \delta')(\gamma \otimes \delta) = \gamma'\gamma \otimes \delta'\delta$ since a similar formula for $\widetilde{\otimes}$ holds. Thus, we have defined a bifunctor $\otimes : \mathbf{V}(X) \times \mathbf{V}(X) \to \mathbf{V}(X)$.

Theorem 3.3.7. *Let X be a scheme and $\mathbf{V}(X)$ the category of standard vector bundles on X.*

(1) The direct sum $\oplus : \mathbf{V}(X) \times \mathbf{V}(X) \to \mathbf{V}(X)$ is strictly associative. In other words, the following diagram commutes (strictly, not up to a natural isomorphism).

$$\begin{tikzcd} \mathbf{V}(X) \times \mathbf{V}(X) \times \mathbf{V}(X) \arrow[r, "\oplus\times 1"] \arrow[d, "1\times\oplus"'] & \mathbf{V}(X) \times \mathbf{V}(X) \arrow[d, "\oplus"] \\ \mathbf{V}(X) \times \mathbf{V}(X) \arrow[r, "\oplus"'] & \mathbf{V}(X) \end{tikzcd}$$

(2) The zero bundle 0 is the strict identity with respect to $\oplus$, In other words, for any $E \in \mathbf{V}(X)$, $0 \oplus E = E \oplus 0 = E$ (identities, not natural isomorphisms), and if $\gamma : E \to F$ is a map of standard vector bundles, then $1_0 \oplus \gamma = \gamma \oplus 1_0 = \gamma$.

(3) If $f : Y \to X$ is a map of schemes, then f^* preserves $\oplus$ and the identity object. In other words, $f^*0 = 0$, and the following diagram commutes

$$\begin{tikzcd} \mathbf{V}(X) \times \mathbf{V}(X) \arrow[r, "\oplus"] \arrow[d, "(f^*, f^*)"'] & \mathbf{V}(X) \arrow[d, "f^*"] \\ \mathbf{V}(Y) \times \mathbf{V}(Y) \arrow[r, "\oplus"'] & \mathbf{V}(Y) \end{tikzcd}$$

Proof. The statements are about the commutativity of various diagrams of functors. Two composite functors are the same when they agree on objects and on morphisms. First, the equality of objects, i.e., standard vector bundles, is shown by proving that they have equal modules of sections and equal restriction maps. Since the modules of sections of standard vector bundles are standard free modules, two of them are the same if and only if they have the same rank. It can be verified easily. So we only need to see if they have the same restriction maps. Suppose (H, E) is a standard vector bundle, and $g : Z \to Y$ is a morphism of $\mathcal{C}_H$. Since $E(Y)$ and $E(Z)$ are standard free modules, the map $E(g) : E(Y) \to E(Z)$ is represented by a matrix (with respect to the standard bases). Suppose K is the sieve associated to F, and g is in $\mathcal{C}_{H \cap K}$. Then $(E \oplus F)(g)$ is represented by the block matrix

$$\begin{pmatrix} E(g) & 0 \\ 0 & F(g) \end{pmatrix}$$

because the standard basis of $(E \oplus F)(W)$ corresponds to those of $E(W)$ and $F(W)$ via the isomorphism

$$\sigma : (a_1, \ldots, a_r, a_{r+1}, \ldots, a_{r+s}) \mapsto ((a_1, \ldots, a_r), (a_{r+1}, \ldots, a_{r+s}))$$

for all relevant objects W. The commutativity of the diagrams on objects follows from this observation. For the commutativity of the first diagram on morphisms, suppose $\gamma : (H, E) \to (H'E')$, $\delta : (K, F) \to (K', F')$, and $\varepsilon : (L, G) \to (L', G')$ are morphisms of standard vector bundles. We need to show $(\gamma \oplus \delta) \oplus \varepsilon = \gamma \oplus (\delta \oplus \varepsilon)$. It suffices to show the commutativity of the following diagram as then the back square shows the equality.

$$\begin{tikzcd}
\xi((E \oplus F) \oplus G) \arrow[rr, "(\gamma\oplus\delta)\oplus\varepsilon"] \arrow[dr, "\cong"] \arrow[dd, "1"'] & & \xi((E' \oplus F') \oplus G') \arrow[dr, "\cong"] \arrow[dd, "1"' near end] & \\
& (\xi E \mathbin{\widetilde{\oplus}} \xi F) \mathbin{\widetilde{\oplus}} \xi G \arrow[rr, "(\gamma \mathbin{\widetilde{\oplus}} \delta) \mathbin{\widetilde{\oplus}} \varepsilon" near start] \arrow[dd, "\alpha"' near start] & & (\xi E' \mathbin{\widetilde{\oplus}} \xi F') \mathbin{\widetilde{\oplus}} \xi G' \arrow[dd, "\alpha"] \\
\xi(E \oplus (F \oplus G)) \arrow[rr, "\gamma\oplus(\delta\oplus\varepsilon)" near end] \arrow[dr, "\cong"] & & \xi(E' \oplus (F' \oplus G')) \arrow[dr, "\cong"] & \\
& \xi E \mathbin{\widetilde{\oplus}} (\xi F \mathbin{\widetilde{\oplus}} \xi G) \arrow[rr, "\gamma \mathbin{\widetilde{\oplus}} (\delta \mathbin{\widetilde{\oplus}} \varepsilon)"'] & & \xi E' \mathbin{\widetilde{\oplus}} (\xi F' \mathbin{\widetilde{\oplus}} \xi G')
\end{tikzcd}$$

In the diagram, the isomorphism α is the associativity isomorphism

$$(((a_1, \ldots, a_r), (b_1, \ldots, b_s)), (c_1, \ldots, c_t)) \mapsto ((a_1, \ldots, a_r), ((b_1, \ldots, b_s), (c_1, \ldots, c_t))),$$

so the front square commutes. The slanted arrows are derived from the isomorphism σ defined by (3.1). Therefore, the left and the right squares commute. The top and bottom squares commute by definition. Therefore, the whole diagram commutes. The property (2) of the theorem is proved similarly. The property (3) follows directly from the definition of f^* since f^*E and $f^*\gamma$ are the same as E and γ everywhere they are defined for any standard vector bundle E and any map γ of standard vector bundles. $\square$

Theorem 3.3.8. *Let X be a scheme and $\mathbf{V}(X)$ the category of standard vector bundles on X.*

(1) The tensor product $\otimes : \mathbf{V}(X) \times \mathbf{V}(X) \to \mathbf{V}(X)$ is strictly associative. In other words, the following diagram commutes (strictly, not up to a natural isomorphism).

$$\begin{tikzcd}
\mathbf{V}(X) \times \mathbf{V}(X) \times \mathbf{V}(X) \arrow[r, "\otimes\times 1"] \arrow[d, "1\times\otimes"'] & \mathbf{V}(X) \times \mathbf{V}(X) \arrow[d, "\otimes"] \\
\mathbf{V}(X) \times \mathbf{V}(X) \arrow[r, "\otimes"'] & \mathbf{V}(X)
\end{tikzcd}$$

(2) The trivial standard line bundle $\mathcal{O}_X$ is the strict identity with respect to $\otimes$, In other words, for any $E \in \mathbf{V}(X)$, $\mathcal{O}_X \otimes E = E \otimes \mathcal{O}_X = E$ (identities, not natural isomorphisms), and if $\gamma : E \to F$ is a map of standard vector bundles, then $1_{\mathcal{O}_X} \otimes \gamma = \gamma \otimes 1_{\mathcal{O}_X} = \gamma$.

(3) Let $\mathbf{L}(X)$ be the category of standard line bundles on X, a full subcategory of $\mathbf{V}(X)$. Then $\mathbf{L}(X)$ is a strict center in the sense that $E \otimes L = L \otimes E$ (identity, not natural isomorphism) for all $E \in \mathbf{V}(X)$

and $L \in \mathbf{L}(X)$, and $\gamma \otimes \beta = \beta \otimes \gamma$ for all morphisms γ of $\mathbf{V}(X)$ and β of $\mathbf{L}(X)$.

(4) If $f : Y \to X$ is a map of schemes, then f^* preserves $\otimes$ and the identity object. In other words, $f^*\mathcal{O}_X = \mathcal{O}_Y$, and the following diagram commutes

$$\begin{tikzcd}
\mathbf{V}(X) \times \mathbf{V}(X) \arrow[r, "\otimes"] \arrow[d, "(f^*, f^*)"'] & \mathbf{V}(X) \arrow[d, "f^*"] \\
\mathbf{V}(Y) \times \mathbf{V}(Y) \arrow[r, "\otimes"'] & \mathbf{V}(Y)
\end{tikzcd}$$

Proof. This theorem is analogous to the previous theorem on direct sums. So the idea of the proof is the same. It is worth to note that if (H, E) and (K, F) are standard vector bundles, and g is a morphism in $\mathcal{C}_{H\cap K}$, then $(E \otimes F)(g)$ is represented by the tensor product of the matrices representing $E(g)$ and $F(g)$ where the tensor product of two matrices A and B is defined to be the following block matrix.

$$\begin{pmatrix} a_{11}B & a_{12}B & \cdots \\ a_{21}B & a_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Therefore, object-wise, $0 \otimes E = E \otimes 0 = 0$ since all involved matrices are empty matrices, (1) is true since $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ for any matrices A,B, and C. (2) is true since $\mathcal{O}_X(g)$ is the 1-by-1 matrix 1, (3) is true since for any standard line bundle L, $L(g)$ is a 1-by-1 matrix, and (4) is true since f^*E is the same as E everywhere it is defined. To prove the equation of (2) on morphisms, suppose $\gamma : (H, E) \to (K, F)$ is a morphism, then there is a commutative diagram

$$\begin{tikzcd}
\xi_H(\mathcal{O}_X \otimes E) \arrow[rr, "1\otimes\gamma"] \arrow[dd, "1"'] \arrow[dr, "\zeta"] & & \xi_K(\mathcal{O}_X \otimes F) \arrow[dd, "1"' near start] \arrow[dr, "\zeta"] & \\
& \xi_{H_X}(\mathcal{O}_X \widetilde{\otimes} \xi_H E) \arrow[rr, "\xi(1 \widetilde{\otimes} \gamma)"' near start] \arrow[dl, "\mu"'] & & \xi_{H_X}(\mathcal{O}_X \widetilde{\otimes} \xi_K F) \arrow[dl, "\mu"] \\
\xi_H E \arrow[rr, "\gamma"'] & & \xi_K F &
\end{tikzcd}$$

In this diagram, ζ is the isomorphism (3.3), which was derived from the isomorphism π^{-1} where π is the isomorphism defined by (3.2), and μ is the isomorphism derived by π. Therefore, the triangles commute. The top and the bottom squares commute by definition. Therefore, the back square commutes, and $1 \otimes \gamma = \gamma$. The commutativity of the diagram in (1) is proved similarly. If $(\gamma, \delta, \varepsilon)$ is a morphism of $\mathbf{V}(X) \times \mathbf{V}(X) \times \mathbf{V}(X)$, then the following diagram similar to the diagram used in the proof of the previous theorem commutes.

$$
\begin{tikzcd}
\xi((E \otimes F) \otimes G) \arrow[rr, "(\gamma\otimes\delta)\otimes\varepsilon"] \arrow[dr, "\cong"'] \arrow[dd, "1"'] & & \xi((E' \otimes F') \otimes G') \arrow[dr, "\cong"] \arrow[dd, "1"' near start] & \\
& \xi(\xi(\xi E \mathbin{\widetilde{\otimes}} \xi F) \mathbin{\widetilde{\otimes}} \xi G) \arrow[rr, "\xi(\xi(\gamma \mathbin{\widetilde{\otimes}} \delta) \mathbin{\widetilde{\otimes}} \varepsilon)" near start] \arrow[dd, "\alpha"' near start] & & \xi(\xi(\xi E' \mathbin{\widetilde{\otimes}} \xi F') \mathbin{\widetilde{\otimes}} \xi G') \arrow[dd, "\alpha"] \\
\xi(E \otimes (F \otimes G)) \arrow[rr, "\gamma\otimes(\delta\otimes\varepsilon)" near end] \arrow[dr, "\cong"'] & & \xi(E' \otimes (F' \otimes G')) \arrow[dr, "\cong"] & \\
& \xi(\xi E \mathbin{\widetilde{\otimes}} \xi(\xi F \mathbin{\widetilde{\otimes}} \xi G)) \arrow[rr, "\xi(\gamma \mathbin{\widetilde{\otimes}} \xi(\delta \mathbin{\widetilde{\otimes}} \varepsilon))"'] & & \xi(\xi E' \mathbin{\widetilde{\otimes}} \xi(\xi F' \mathbin{\widetilde{\otimes}} \xi G'))
\end{tikzcd}
$$

Note that for the commutativity of the left and the right squares, we use the fact that $\pi(\pi(u, v), w) = \pi(u, \pi(v, w))$ for any three vectors u, v, and w. The property (3) follows from the commutativity of the next diagram.

$$
\begin{tikzcd}
\xi(E \otimes L) \arrow[rr, "\gamma\otimes\beta"] \arrow[dr, "\cong"'] \arrow[dd, "1"'] & & \xi(E' \otimes L') \arrow[dr, "\cong"] \arrow[dd, "1"' near end] & \\
& \xi(\xi E \mathbin{\widetilde{\otimes}} \xi L) \arrow[rr, "\xi(\gamma \mathbin{\widetilde{\otimes}} \beta)" near start] \arrow[dd, "\tau"' near start] & & \xi(\xi E' \mathbin{\widetilde{\otimes}} \xi L') \arrow[dd, "\tau"] \\
\xi(L \otimes E) \arrow[rr, "\beta\otimes\gamma"' near end] \arrow[dr, "\cong"'] & & \xi(L' \otimes E') \arrow[dr, "\cong"] & \\
& \xi(\xi L \mathbin{\widetilde{\otimes}} \xi E) \arrow[rr, "\xi(\beta \mathbin{\widetilde{\otimes}} \gamma)"'] & & \xi(\xi L' \otimes \xi E')
\end{tikzcd}
$$

For the commutativity of the left and the right squares, we need the fact that $\pi(u, v) = \pi(v, u)$ if u or v is a 1-dimensional vector. The property (5) follows from the fact that f^*E and $f^*\gamma$ are the same as E and γ everywhere they are defined for any standard vector bundle E and any map γ of standard vector bundles. $\square$

Remark 3.3.9. The reason the tensor product is not strictly commutative in general is that for a commutative ring A, a choice needs to be made to define an isomorphism $A^r \otimes A^s \to A^{rs}$, and no choice is symmetric unless $r \leq 1$ or $s \leq 1$. For example, if a, b, c, and d are elements of A, then $(a, b) \otimes (c, d) = \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix}$ and $(c, d) \otimes (a, b) = \begin{pmatrix} ac & bc \\ ad & bd \end{pmatrix}$. Thus, for an arbitrary isomorphism $f : A^r \otimes A^s \to A^{rs}$, we cannot expect $f((a, b) \otimes (c, d))$ and $f((c, d) \otimes (a, b))$ to be equal since $ad \neq bc$ in general.

Theorem 3.3.10. *Let X be a scheme in Sm/S and $\mathbf{V}(X)$ the category of standard vector bundles on X.*

(1) $\mathbf{V}(X)$ is a small exact category.

(2) Let $\mathcal{P}(X)$ be the category of locally free $\mathcal{O}_X$-modules of finite rank. There are exact functors $\Phi : \mathbf{V}(X) \to \mathcal{P}(X)$ and $\Psi : \mathcal{P}(X) \to \mathbf{V}(X)$ that are equivalences of categories.

(3) If $f : Y \to X$ is a map of schemes, then $f^* : \mathbf{V}(X) \to \mathbf{V}(Y)$ is an exact functor. If $g : Z \to Y$ is another map of schemes, then $g^* f^* = (fg)^*$ as functors $\mathbf{V}(X) \to \mathbf{V}(Z)$. (It is an equality, not simply a natural isomorphism.)

(4) The tensor product $\otimes : \mathbf{V}(X) \times \mathbf{V}(X) \to \mathbf{V}(X)$ is a biexact pairing, in other words, for any $E \in \mathbf{V}(X)$, $0 \otimes E = E \otimes 0 = 0$, and if $\mathcal{S}$ is a short exact sequence of $\mathbf{V}(X)$, then so are $\mathcal{S} \otimes E$ and $E \otimes \mathcal{S}$.

Proof. The category $\mathbf{V}(X)$ is small because $(Sch/X)_{Zar}$ is small and the values of a standard vector bundle at objects are standard free modules. It will be shown to be an exact category later.

Define a functor $\Phi : \mathbf{V}(X) \to \mathcal{P}(X)$ as follows. Suppose E is a standard vector bundle on X with the associated sieve H. Define $\Phi E = \xi_H E|_X$, the restriction of the sheafification of E to the small Zariski site of X. There is a Zariski covering $\{U_i \xrightarrow{f_i} X\}$ with f_i in $\mathcal{C}_H$ since H belongs to $(Sch/X)_{Zar}$. For each i, we have an isomorphism $\mathcal{O}^n_{U_i} \to f_i^* E$. Restricting it to the small Zariski site of U_i, we get an isomorphism $\mathcal{O}^n_{U_i} \to (f_i^* E)|_{U_i} = E|_{U_i}$. Applying the sheafification functor, we get an isomorphism $\mathcal{O}^n_{U_i} \to \xi(E|_{U_i})$. By Proposition 3.2.5, $\Phi E|_{U_i} = (\xi_H E|_X)|_{U_i} = \xi(E|_{U_i})$ Hence, ΦE is indeed a locally free sheaf. If (K, F) is another standard vector bundle on X and $\alpha : (H, E) \to (K, F)$ is a morphism, that is, a map $\alpha : \xi_H E \to \xi_K F$ of sheaves on $(Sch/X)_{Zar}$, then $\Phi(\alpha)$ is defined to be the induced map $\alpha|_X : \xi_H E|_X \to \xi_K F|_X$ of sheaves on X. This assignment respects the composition of morphisms since the restriction $-|_X$ is a functor. Hence Φ is a functor.

Next, we define the inverse $\Psi : \mathcal{P}(X) \to \mathbf{V}(X)$. If $\mathcal{E}$ is a locally free sheaf, then define $\Psi\mathcal{E} = S^{\varphi}_H B\mathcal{E}|_H$. (See Lemma 3.3.3 and Corollary 3.3.4.) Note that a choice of a sieve H and a collection of isomorphisms φ needs to be made for each $\mathcal{E}$. Suppose that $\mathcal{F}$ is another locally free sheaf and that $\beta : \mathcal{E} \to \mathcal{F}$ is a map of sheaves. If the sieve and the isomorphisms associated to $\mathcal{F}$ are K and ψ, then $\Psi\mathcal{F} = S^{\psi}_K B\mathcal{E}|_K$, and there are isomorphisms of sheaves $\gamma_{\mathcal{E}} : \xi_H \Psi\mathcal{E} \to B\mathcal{E}$ and $\gamma_{\mathcal{F}} : \xi_K \Psi\mathcal{F} \to B\mathcal{F}$ by Corollary 3.3.4. Using them, define $\Psi\beta = \gamma_{\mathcal{F}}^{-1}(B\beta)\gamma_{\mathcal{E}}$.

$$\Psi\beta : \xi_H \Psi\mathcal{E} \xrightarrow{\gamma_{\mathcal{E}}} B\mathcal{E} \xrightarrow{B\beta} B\mathcal{F} \xrightarrow{\gamma_{\mathcal{F}}^{-1}} \xi_K \Psi\mathcal{F}$$

It was defined in such a way that the following diagram commutes, so that we may identify the sheafification of $\Psi\mathcal{E}$ with $B\mathcal{E}$ intrinsically without reference to the choice of H and φ.

$$\begin{tikzcd}
\xi_H \Psi\mathcal{E} \arrow[r, "\gamma_{\mathcal{E}}"] \arrow[d, "\Psi\alpha"'] & B\mathcal{E} \arrow[d, "B\alpha"] \\
\xi_K \Psi\mathcal{F} \arrow[r, "\gamma_{\mathcal{F}}"'] & B\mathcal{F}
\end{tikzcd} \tag{3.4}$$

This assignment respects the composition of morphisms since B is a functor. Therefore, Ψ is a functor.

Now we prove that Φ and Ψ are inverses to each other. Suppose $\mathcal{E} \in \mathcal{P}(X)$. By Lemma 3.2.6, there is an isomorphism

$$\Phi\Psi\mathcal{E} = (\xi_H \Psi\mathcal{E})|_X \xrightarrow[\cong]{\gamma_{\mathcal{E}}|_X} B\mathcal{E}|_X \cong 1_X^* \mathcal{E} \cong \mathcal{E}.$$

This isomorphism is natural in $\mathcal{E}$ since for a morphism $\alpha : \mathcal{E} \to \mathcal{F}$ of $\mathcal{P}(X)$, the following diagram commutes. (The left square commutes by definition, the middle by (3.4), and the right by the naturality of the isomorphism of Lemma 3.2.6.)

$$\begin{tikzcd}
\Phi\Psi\mathcal{E} \arrow[r, equal] \arrow[d, "\Phi\Psi\alpha"'] & \xi_H \Psi\mathcal{E}|_X \arrow[r, "\gamma_{\mathcal{E}}|_X"] \arrow[d, "(\Psi\alpha)|_X"'] & B\mathcal{E}|_X \arrow[r, "\cong"] \arrow[d, "(B\alpha)|_X"] & \mathcal{E} \arrow[d, "\alpha"] \\
\Phi\Psi\mathcal{F} \arrow[r, equal] & \xi_H \Psi\mathcal{F}|_X \arrow[r, "\gamma_{\mathcal{F}}|_X"'] & B\mathcal{F}|_X \arrow[r, "\cong"'] & \mathcal{F}
\end{tikzcd}$$

Therefore, $\Phi\Psi$ is naturally isomorphic to the identity functor on $\mathcal{P}(X)$. Conversely, suppose E is an H-vector bundle, and suppose $\Psi\Phi E$ turns out to be a K-vector bundle. The isomorphism

$$\gamma_{\Phi\mathcal{E}} : \xi_K \Psi\Phi E \to B\Phi E = B(\xi_H E|_X)$$

of Corollary 3.3.4 is natural in E by the diagram (3.4). In addition, there is a natural isomorphism $B(\xi_H E|_X) \cong \xi_H E$ by Lemma 3.2.7 and Lemma 3.3.6. Composing them together, we obtain a natural isomorphism $\xi_K \Psi\Phi E \cong \xi_H E$. Therefore, $\Psi\Phi$ is naturally isomorphic to the identity functor. This proves the equivalence of $\mathcal{P}(X)$ and $\mathbf{V}(X)$.

The category $\mathbf{V}(X)$ of standard vector bundles is additive with $\oplus$ as the biproduct operation. The functors Φ and Ψ are additive since

$$\Phi(E \oplus F) = \xi_{H\cap K}(E \oplus F)|_X \cong \xi_H E|_X \oplus \xi_K F|_X = \Phi E \oplus \Phi F,$$
$$\Psi(\mathcal{E} \oplus \mathcal{F}) \cong B(\mathcal{E} \oplus \mathcal{F}) \cong B\mathcal{E} \oplus B\mathcal{F} \cong \Psi\mathcal{E} \oplus \Psi\mathcal{F},$$

and projections and injections are preserved. The category $\mathcal{P}(X)$ is well known to be an exact category, (as a full subcategory of the abelian category of $\mathcal{O}_X$-modules closed under extensions,) and $\mathbf{V}(X)$ is equivalent to $\mathcal{P}(X)$. Therefore, $\mathbf{V}(X)$ can be given the structure of an exact category such that the equivalences Φ and Ψ become exact functors by transporting the notion of exactness from $\mathcal{P}(X)$ to $\mathbf{V}(X)$, that is, a sequence $0 \to E \to F \to G \to 0$ of standard vector bundles is defined to be exact if and only if $0 \to \Phi E \to \Phi F \to \Phi G \to 0$ is exact.

To prove the third property of the theorem, suppose $f : Y \to X$ is a map of schemes. For any standard vector bundle (H, E) in $\mathbf{V}(X)$, we have a natural isomorphism

$$\Phi f^* E = \xi_{f^*H} f^* E|_Y = f^* \xi_H E|_Y = \xi_H E|_Y \xleftarrow{\cong} f^*(\xi_H E|_X) = f^* \Phi E$$

by the definition of Φ, Lemma 3.2.11, and Lemma 3.2.8. If $0 \to E \to F \to G \to 0$ is a short exact sequence in $\mathbf{V}(X)$, then the sequence $0 \to \Phi E \to \Phi F \to \Phi G \to 0$ is exact. Hence $0 \to f^*\Phi E \to f^*\Phi F \to f^*\Phi G \to 0$ is exact, and so is $0 \to \Phi f^* E \to \Phi f^* F \to \Phi f^* G \to 0$. Therefore, $0 \to f^*E \to f^*F \to f^*G \to 0$ is an exact sequence in $\mathbf{V}(Y)$. This proves that $f^* : \mathbf{V}(X) \to \mathbf{V}(Y)$ is an exact functor. If $g : Z \to Y$ is another map of schemes, then $(fg)^* = g^* f^*$ by Proposition 3.3.5.

Finally, we prove that $\otimes$ is biexact. First note that for any any scheme Y over X and any standard vector bundles (H, E) and (K, F) on X, there is an isomorphisms.

$$\begin{aligned}
\xi_{H\cap K}(E \otimes F)|_Y &\cong \xi_{H_X}(\xi_H E \,\widetilde{\otimes}\, \xi_K F)|_Y && (3.5)\\
&= \xi((\xi_H E \,\widetilde{\otimes}\, \xi_K F)|_Y) && (3.6)\\
&= \xi(\xi_H E|_Y \,\widetilde{\otimes}\, \xi_K F|_Y) && (3.7)\\
&= \xi_H E|_Y \otimes_{\mathcal{O}_Y} \xi_K F|_Y && (3.8)
\end{aligned}$$

We used the isomorphism (3.3) for (3.5), Proposition 3.2.5 for (3.6) and the definition of the tensor product of $\mathcal{O}_Y$-modules for (3.8). If $\alpha : (H, E) \to (H', E')$ and $\beta : (K, F) \to (K', F')$ are maps of standard vector bundles, then the map $(\alpha \otimes \beta)|_Y$ corresponds to the map $\alpha|_Y \otimes \beta|_Y$. Suppose $\mathcal{S}$ is a short exact sequence below,

$$0 \to (H, E) \xrightarrow{\alpha} (K, F) \xrightarrow{\beta} (L, G) \to 0$$

and (M, D) is a standard vector bundle. It is enough to prove the following sequence $D \otimes \mathcal{S}$ is exact, the other being similar.

$$0 \to \xi_{M\cap H}(D\otimes E) \xrightarrow{1\otimes\alpha} \xi_{M\cap K}(D\otimes F) \xrightarrow{1\otimes\beta} \xi_{M\cap L}(D\otimes G) \to 0$$

By the definition of exactness for standard vector bundles, we need to prove that the following sequence of $\mathcal{O}_X$-modules is exact.

$$0 \to \xi_{M\cap H}(D\otimes E)|_X \xrightarrow{(1\otimes\alpha)|_X} \xi_{M\cap K}(D\otimes F)|_X \xrightarrow{(1\otimes\beta)|_X} \xi_{M\cap L}(D\otimes G)|_X \to 0$$

But it is isomorphic to the sequence

$$0 \to \xi_M D|_X \otimes \xi_H E|_X \xrightarrow{1\otimes\alpha|_X} \xi_M D|_X \otimes \xi_K F|_X \xrightarrow{1\otimes\beta|_X} \xi_M D|_X \otimes \xi_L G|_X \to 0,$$

which is an exact sequence of locally free $\mathcal{O}_X$-modules since $\otimes$ is a biexact pairing on the category of locally free $\mathcal{O}_X$-modules.

□

3.3.3 Twisted sheaf as a standard line bundle

In this section, we discuss the twisted sheaf $\mathcal{O}(n)$ on a projective space. There could be many standard vector bundles that correspond to $\mathcal{O}(n)$. But for our purposes, we need the one that behaves well under pullbacks and base change. In Theorem 3.3.10, the way we constructed a standard vector bundle from an ordinary vector bundle was to use Corollary 3.3.4 after choosing a covering that trivializes the vector bundle and an isomorphism to a standard free module for each scheme factoring through one of the open covers. We will show how the choices can be made universally for $\mathcal{O}(n)$ to suit our needs.

Let $\mathbb{P}^r_X = X \times_{\mathbb{Z}} \operatorname{Proj} \mathbb{Z}[x_0, x_1, \dots, x_r]$. It is covered by $U_0, U_1, \dots, U_r$ where

$$U_k = X \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{Z}\left[\frac{x_0}{x_k}, \frac{x_1}{x_k}, \dots, \frac{\widehat{x_k}}{x_k}, \dots, \frac{x_r}{x_k}\right] \quad \text{for} \quad k = 0, 1, \dots, r.$$

Let $i_k : U_k \to \mathbb{P}^r_X$ be the inclusions, and let H be the sieve generated by them. For each k, there is a map $x_k^n : \mathcal{O}_{U_k} \to i_k^* \mathcal{O}_{\mathbb{P}^r_X}(n)$ of sheaves on $(U_k)_{zar}$ defined by multiplication by x_k^n. It is an isomorphism because x_k is invertible in U_k. Suppose $Y \xrightarrow{h} \mathbb{P}^r_X$ is an object of $\mathcal{C}_H$. Then h factors as

$$Y \xrightarrow{h_k} U_k \xrightarrow{i_k} \mathbb{P}^r_X$$

for some k. We choose the largest such k. (One could make many different choices here, but our choice is made for Lemma 3.3.11 to work.) Then there is an isomorphism

$$\alpha_h : \mathcal{O}_Y \cong h_k^*\mathcal{O}_{U_k} \xrightarrow[\cong]{h_k^*(x_k^n)} h_k^* i_k^* \mathcal{O}_{\mathbb{P}^r_X}(n) \underset{\cong}{\longrightarrow} h^*\mathcal{O}_{\mathbb{P}^r_X}(n)$$

Define φ_h to be the following composite map as in Corollary 3.3.4.

$$\mathcal{O}_Y \cong B\mathcal{O}_Y \xrightarrow[\cong]{B\alpha_h} Bh^*\mathcal{O}_{\mathbb{P}^r_X}(n) \underset{\cong}{\longrightarrow} h^*(B\mathcal{O}_{\mathbb{P}^r_X}(n)|_H).$$

With these choices of H and φ_h's, define $\mathcal{O}_{\mathbb{P}^r_X}(n)$ as a standard line bundle to be $S^{\varphi}_H B\mathcal{O}_{\mathbb{P}^r_X}(n)|_H$.

Lemma 3.3.11. *Suppose $i_\infty : S \to \mathbb{P}^1_S$ is the inclusion of the point $\infty = [0 : 1]$. Then $i_\infty^*\mathcal{O}_{\mathbb{P}^1_S}(n) = \mathcal{O}_S$ for any n.*

Proof. The projective line $\mathbb{P}^1_S$ is covered by two affine lines U_0 and U_1 as above. Let H be the sieve generated by them. For any scheme X over S, the composite map $X \to S \xrightarrow{i_\infty} \mathbb{P}^1_S$ factors through U_1. Hence $i_\infty^* H = H_S$, and $i_\infty^*\mathcal{O}_{\mathbb{P}^1_S}(n)$ is defined uniformly by the isomorphism x_1^n identifying $\mathcal{O}_{U_1}$ with $i_1^*\mathcal{O}_{\mathbb{P}^1_S}(n)$. Therefore, $i_\infty^*\mathcal{O}_{\mathbb{P}^1_S}(n) = \mathcal{O}_S$. $\square$

Lemma 3.3.12. *Suppose $f : Y \to X$ be a map of schemes. Let $g : \mathbb{P}^r_Y \to \mathbb{P}^r_X$ be the induced map $f \times 1$. Then $g^*\mathcal{O}_{\mathbb{P}^r_X}(n) = \mathcal{O}_{\mathbb{P}^r_Y}(n)$ (equality, not simply natural isomorphism) for every n.*

Proof. Let H be the sieve on $\mathbb{P}^r_X$ generated by the covering $\{U_{X,k} \to \mathbb{P}^r_X\}$ described above, and let K be the sieve on $\mathbb{P}^r_Y$ generated by the analogous covering $\{U_{Y,k} \to \mathbb{P}^r_Y\}$ of $\mathbb{P}^r_Y$. For each $0 \le k \le r$, a map $h : Z \to \mathbb{P}^r_Y$ factors through $U_{Y,k}$ if and only if the composite gh factors through $U_{X,k}$ since $U_{Y,k} \cong U_{X,k} \times_{\mathbb{P}^r_X} \mathbb{P}^r_Y$.

$$\begin{tikzcd}
Z \arrow[dr] \arrow[drr, dashed] \arrow[ddr, "h"'] & & \\
& U_{Y,k} \arrow[r] \arrow[d] & U_{X,k} \arrow[d] \\
& \mathbb{P}^r_Y \arrow[r, "g"'] & \mathbb{P}^r_X
\end{tikzcd}
\qquad
\begin{tikzcd}
Z \arrow[dr, dashed] \arrow[drr] \arrow[ddr, "h"'] & & \\
& U_{Y,k} \arrow[r] \arrow[d] & U_{X,k} \arrow[d] \\
& \mathbb{P}^r_Y \arrow[r, "g"'] & \mathbb{P}^r_X
\end{tikzcd}$$

Therefore, $g^*H = K$. Moreover, for each object $h : Z \to \mathbb{P}^r_Y$ of $\mathcal{C}_K$, the standardizing maps φ_h for $\mathcal{O}_{\mathbb{P}^r_Y}(n)(Z)$ and $g^*\mathcal{O}_{\mathbb{P}^r_X}(n)(Z)$ are defined by multiplication by x_k^n, both with the same k. Therefore, $g^*\mathcal{O}_{\mathbb{P}^r_X}(n) = \mathcal{O}_{\mathbb{P}^r_Y}(n)$ as standard line bundles. $\square$

Chapter 4

The Gillet-Grayson construction of K-theory

In this section, we review the G-construction of K-theory from [7, 8]. The G-construction of an exact category $\mathcal{E}$ with a chosen zero object is homotopy equivalent to the loop space of the Waldhausen $S.$-construction of $\mathcal{E}$, and the iteration of the G-construction does not change the homotopy type. The product in K-theory is easily described using iterated G-constructions,

$$G^p\mathcal{E} \wedge G^q\mathcal{E} \to G^{p+q}\mathcal{E},$$

and we use this property to construct the motivic K-theory spectrum in the next chapter. Nothing is original in this chapter.

4.1 The G-construction

Let Δ be the category of finite nonempty ordered sets $\underline{n} = \{0 < 1 < \cdots < n\}$ with order-preserving maps. Denote the category of sets by **Set**. For each $\underline{n}$, define $\gamma(\underline{n})$ to be the poset with two incomparable minimal elements attached called $+$ and $-$. It can be described as in the following diagram.

$$\begin{tikzcd}
+ \arrow[dr] & & & & \\
& 0 \arrow[r] & 1 \arrow[r] & \cdots \arrow[r] & n \\
- \arrow[ur] & & & &
\end{tikzcd} \tag{4.1}$$

Then γ defines a functor from Δ to the category of posets. A map $\alpha : \underline{m} \to \underline{n}$ is sent to $\gamma\alpha : \gamma(\underline{m}) \to \gamma(\underline{n})$ defined by $\gamma\alpha(+) = +$, $\gamma\alpha(-) = -$ and $\gamma\alpha(i) = \alpha(i)$ for $0 \le i \le m$.

Suppose $\mathcal{E}$ is an exact category with a chosen zero object 0, and P is a poset. An object of the arrow category $\mathrm{Ar}(P)$ will be denoted by j/i if $i \le j$. A functor $M : \mathrm{Ar}(P) \to \mathcal{E}$ is said to be *exact* if for all $i \le j \le k$, $0 \to M(j/i) \to M(k/i) \to M(k/j) \to 0$ is a short exact sequence in $\mathcal{E}$ and $M(i/i) = 0$. If

$P_1, P_2, \ldots, P_r$ are posets, a functor $M : \prod_{i=1}^{r} \mathrm{Ar}(P_i) \to \mathcal{E}$ is said to be *multiexact* (or *biexact* when $r = 2$) if it is exact on each factor of the product.

Definition 4.1.1 ([7], [8])**.** Let $\mathcal{E}$ be a small exact category with a chosen 0 object. We define a simplicial set $G.\mathcal{E}$ by letting it send $\underline{n} \in \Delta$ to the set of exact functors $\Gamma(\underline{n}) \to \mathcal{E}$ where $\Gamma(\underline{n}) = \mathrm{Ar}(\gamma(\underline{n}))$.

We may consider Γ as a functor from Δ to the category of categories. (A map $\alpha : \underline{m} \to \underline{n}$ defines a functor $\Gamma(\alpha) : \Gamma(\underline{m}) \to \Gamma(\underline{n})$ sending an arrow j/i to $\gamma\alpha(j)/\gamma\alpha(i)$.) We will also use the same notation Γ for the functor from Δ^r to the category of categories defined by $\Gamma(A) = \prod_{i=1}^{r} \mathrm{Ar}(\gamma(\underline{a_i}))$ where $A = (\underline{a_1}, \ldots, \underline{a_r}) \in \Delta^r$.

Theorem 4.1.2 ([7])**.** *There is a homotopy equivalence $|G.\mathcal{E}| \xrightarrow{\sim} \Omega|S.\mathcal{E}|$ where $S.\mathcal{E}$ is Waldhausen's $S.$-construction. In particular,*

$$\pi_i(G.\mathcal{E}) \cong K_i(\mathcal{E}) \quad \textit{for } i \geq 0.$$

The G-construction can be iterated and used to describe the product operation in K-theory. We define $\mathcal{G}.\mathcal{E}$ to be the simplicial category such that $\mathcal{G}_n\mathcal{E}$ is the full subcategory of the category of functors $\Gamma(\underline{n}) \to \mathcal{E}$ whose objects are exact functors. Then $G_n\mathcal{E}$ is the set of objects of $\mathcal{G}_n\mathcal{E}$. We say that a sequence of exact functors $0 \to M' \to M \to M'' \to 0$ in $\mathcal{G}_n\mathcal{E}$ is *exact* if its evaluation at each object of $\Gamma(\underline{n})$ is exact. This makes $\mathcal{G}_n\mathcal{E}$ into an exact category. So we can apply $G.$ to $\mathcal{G}_n\mathcal{E}$, and this defines a bisimplicial set $G.\mathcal{G}.\mathcal{E} : \Delta^2 \to \mathbf{Set}$ such that $G_m\mathcal{G}_n\mathcal{E}$ is the set of exact functors $\Gamma(\underline{m}) \to \mathcal{G}_n\mathcal{E}$ of exact functors, or equivalently, biexact functors $\Gamma(\underline{m}, \underline{n}) \to \mathcal{E}$.

Lemma 4.1.3 ([7])**.** *There is a homotopy equivalence $G.\mathcal{E} \to \mathrm{diag}(G.\mathcal{G}.\mathcal{E})$ where* diag *is the diagonalization functor from multisimplicial sets to simplicial sets.*

The lemma allows us to iterate the G-construction without altering the homotopy type of the space.

Definition 4.1.4. Let $\mathcal{E}$ be a small exact category with a chosen 0 object and $r > 0$ an integer. We define a multisimplicial set $G^r\mathcal{E} : \Delta^r \to \mathbf{Set}$ by letting it send $A \in \Delta^r$ to the set of multiexact functors $\Gamma(A) \to \mathcal{E}$. We call it *the iterated G-construction of $\mathcal{E}$.*

A 0-simplex of $G\mathcal{E}$ consists of a pair of objects (P, Q) for some objects P and Q in $\mathcal{E}$, where P and Q correspond to $+$ and $-$, respectively in the diagram (4.1), and $G^2\mathcal{E}$ consists of quadruples of objects arranged in a square $\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$. In general, a vertex of $G^r\mathcal{E}$ is 2^r objects in $\mathcal{E}$ arranged in an r-dimensional cube. An n-simplex of $G\mathcal{E}$ consists of a pair $([P], [Q])$ where $[P]$ and $[Q]$ are filtrations of length n, $0 = P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n = P$, $0 = Q_0 \subseteq Q_1 \subseteq \cdots \subseteq Q_n = Q$ with compatible subquotients as specified by the exactness condition of the definition. In general, a simplex of $G^r\mathcal{E}$ is represented by 2^r objects with some multi-filtrations and compatibility conditions.

The definition of $G^r\mathcal{E}$ is functorial in $\mathcal{E}$. If $\mathcal{E}'$ is another exact category and $\varphi : \mathcal{E} \to \mathcal{E}'$ is an exact functor, then we can define $G^r\varphi : G^r\mathcal{E} \to G^r\mathcal{E}'$ by sending an exact functor $M : \Gamma(A) \to \mathcal{E}$ to the composite map $\varphi M : \Gamma(A) \to \mathcal{E} \to \mathcal{E}'$. If $\psi : \mathcal{E}' \to \mathcal{E}''$ is another exact functor, then

$$G^r(\psi \circ \varphi) = G^r\psi \circ G^r\varphi \tag{4.2}$$

We can identify $G^r\mathcal{E}$ with $G.\mathcal{G}.\cdots\mathcal{G}.\mathcal{E}$, therefore,

Lemma 4.1.5. *For a small exact category $\mathcal{E}$ with a chosen 0 object,*

$$|G^1\mathcal{E}| \simeq |G^2\mathcal{E}| \simeq \cdots \simeq |G^r\mathcal{E}| \simeq \cdots$$

and $\pi_i(G^r\mathcal{E}) \cong K_i(\mathcal{E})$ for all $i \geq 0$ and $r \geq 1$.

4.2 The product in K-theory described by the G-construction

Suppose $\mathcal{E}$ is a small exact category with a biexact pairing $\otimes : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$. Then it induces a map $G\mathcal{E} \times G\mathcal{E} \to G^2\mathcal{E}$. For each $n, m \geq 0$, the map $G_n\mathcal{E} \times G_m\mathcal{E} \to G_nG_m\mathcal{E}$ sends a pair of exact functors (M, N) to the biexact functor $M \otimes N$ sending an arrow $(j/i, b/a)$ to $M(j/i) \otimes N(b/a)$, and an arrow of arrows $(f, g) : (j/i, b/a) \to (j'/i', b'/a')$ to $Mf \otimes Ng : M(j/i) \otimes N(b/a) \to M(j'/i') \otimes N(b'/a')$. The functor $M \otimes N$ is biexact because we have a commutative diagram with exact rows and exact columns for each $i \leq j \leq k$ and $a \leq b \leq c$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M(j/i) \otimes N(b/a) \arrow[r] \arrow[d] & M(k/i) \otimes N(b/a) \arrow[r] \arrow[d] & M(k/i) \otimes N(b/a) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M(j/i) \otimes N(c/a) \arrow[r] \arrow[d] & M(k/i) \otimes N(c/a) \arrow[r] \arrow[d] & M(k/i) \otimes N(c/a) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M(j/i) \otimes N(c/b) \arrow[r] \arrow[d] & M(k/i) \otimes N(c/b) \arrow[r] \arrow[d] & M(k/i) \otimes N(c/b) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

In general, for each $p, q \geq 1$, we can define a map $\otimes : G^p\mathcal{E} \times G^q\mathcal{E} \to G^{p+q}\mathcal{E}$ similarly. For $A \in \Delta^p, B \in \Delta^q$, and multiexact functors $M \in G^p\mathcal{E}(A)$ and $N \in G^q\mathcal{E}(B)$, their product $M \otimes N \in G^{p+q}\mathcal{E}(A, B)$ is defined by sending an object $(\alpha, \beta) \in \Gamma(A) \times \Gamma(B) = \Gamma(A, B)$ to $M\alpha \otimes N\beta$ and sending a morphism $(f, g) : (\alpha', \beta') \to (\alpha, \beta)$ to the map $Mf \otimes Ng : M\alpha \otimes N\beta \to M\alpha' \otimes N\beta'$.

Definition 4.2.1. A bifunctor $\otimes : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ on a category $\mathcal{E}$ is said to be *strictly associative* if the following diagram commutes (strictly, not up to a natural isomorphism).

$$\begin{tikzcd}
\mathcal{E} \times \mathcal{E} \times \mathcal{E} \arrow[r, "\otimes\times 1"] \arrow[d, "1\times\otimes"'] & \mathcal{E} \times \mathcal{E} \arrow[d, "\otimes"] \\
\mathcal{E} \times \mathcal{E} \arrow[r, "\otimes"'] & \mathcal{E}
\end{tikzcd}$$

An equivalent condition is that $(E \otimes F) \otimes G = E \otimes (F \otimes G)$ for all objects E, F, and G of $\mathcal{E}$, and $(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma)$ for all morphisms α, β, and γ of $\mathcal{E}$. For example, the tensor product operation on the category $\mathbf{V}(X)$ of standard vector bundles on a scheme X is strictly associative, (see Theorem 3.3.8 (1),) but the tensor product on the category of modules over a commutative ring is not strictly associative since the diagram commutes only up to a natural isomorphism.

Proposition 4.2.2. *Let $\mathcal{E}$ be an exact category with a chosen 0 object. Suppose $\otimes$ is a strictly associative biexact pairing on $\mathcal{E}$. Then the induced maps $\otimes : G^p\mathcal{E} \times G^q\mathcal{E} \to G^{p+q}\mathcal{E}$ defined above for $p, q \geq 1$ are associative, in the sense that the following diagram commutes for all $p, q, r \geq 1$.*

$$\begin{tikzcd}
G^p\mathcal{E} \times G^q\mathcal{E} \times G^r\mathcal{E} \arrow[r, "\otimes\times 1"] \arrow[d, "1\times\otimes"'] & G^{p+q}\mathcal{E} \times G^r\mathcal{E} \arrow[d, "\otimes"] \\
G^p\mathcal{E} \times G^{q+r}\mathcal{E} \arrow[r, "\otimes"'] & G^{p+q+r}\mathcal{E}
\end{tikzcd}$$

Proof. Suppose (M, N, L) is a simplex of $G^p\mathcal{E} \times G^q\mathcal{E} \times G^r\mathcal{E}$, that is, M, N, and L are multiexact functors in $G^p\mathcal{E}(A), G^q\mathcal{E}(B)$, and $G^r\mathcal{E}(C)$, respectively, for some $A \in \Delta^p, B \in \Delta^q, C \in \Delta^r$. We need to show that $(M \otimes N) \otimes L = M \otimes (N \otimes L)$, which is an equality of functors. To prove the equality on objects, let (α, β, γ) be an object of $\Gamma(A) \times \Gamma(B) \times \Gamma(C)$. Then

$$((M \otimes N) \otimes L)(\alpha, \beta, \gamma) = (M\alpha \otimes N\beta) \otimes L\gamma = M\alpha \otimes (N\beta \otimes L\gamma) = (M \otimes (N \otimes L))(\alpha, \beta, \gamma)$$

by the strict associativity of the pairing on $\mathcal{E}$. Similarly, if $(f, g, h) : (\alpha, \beta, \gamma) \to (\alpha', \beta', \gamma')$ is a morphism, then

$$((M \otimes N) \otimes L)(f, g, h) = (Mf \otimes Ng) \otimes Nh = Mf \otimes (Ng \otimes Nh) = (M \otimes (N \otimes L))(f, g, h).$$

$\square$

Proposition 4.2.3. *Suppose $\mathcal{E}$ and $\mathcal{F}$ are exact categories with chosen 0 objects, and suppose they have biexact pairings*

$$\otimes : \mathcal{E} \times \mathcal{E} \to \mathcal{E}, \qquad \otimes : \mathcal{F} \times \mathcal{F} \to \mathcal{F}.$$

Suppose, furthermore, that there is a functor $\varphi : \mathcal{E} \to \mathcal{F}$ preserving the pairings in the sense that the following diagram commutes (strictly, not up to a natural isomorphism).

$$\begin{tikzcd}
\mathcal{E} \times \mathcal{E} \arrow[r, "\otimes"] \arrow[d, "(\varphi,\varphi)"'] & \mathcal{E} \arrow[d, "\varphi"] \\
\mathcal{F} \times \mathcal{F} \arrow[r, "\otimes"'] & \mathcal{F}
\end{tikzcd}$$

Then the following diagram commutes for every $p, q \geq 1$.

$$\begin{tikzcd}
G^p\mathcal{E} \times G^q\mathcal{E} \arrow[r, "\otimes"] \arrow[d, "G^p\varphi \times G^q\varphi"'] & G^{p+q}\mathcal{E} \arrow[d, "G^{p+q}\varphi"] \\
G^p\mathcal{F} \times G^q\mathcal{F} \arrow[r, "\otimes"'] & G^{p+q}\mathcal{F}
\end{tikzcd}$$

Proof. The proof is similar to the proof of Proposition 4.2.2. Suppose $A \in \Delta^p$, $B \in \Delta^q$, and suppose that $M \in G^p\mathcal{E}(A)$ and $N \in G^q\mathcal{E}(B)$ are multiexact functors. Then for any object (α, β) of $\Gamma(A) \times \Gamma(B)$,

$$(G^{p+q}\varphi(M \otimes N))(\alpha, \beta) = \varphi(M\alpha \otimes N\beta) = (\varphi M\alpha) \otimes (\varphi N\beta) = (G^p\varphi M \otimes G^q\varphi N)(\alpha, \beta),$$

and for any morphism (f, g) of $\Gamma(A) \times \Gamma(B)$,

$$(G^{p+q}\varphi(M \otimes N))(f, g) = \varphi(Mf \otimes Ng) = (\varphi Mf) \otimes (\varphi Ng) = (G^p\varphi M \otimes G^q\varphi N)(f, g).$$

$\square$

We can think of $G^\bullet\mathcal{E}$ as a pointed multisimplicial set with the zero functor 0 as the base point. Since $0 \otimes -$ and $- \otimes 0$ are zero functors, we have a pairing $G^p\mathcal{E} \wedge G^q\mathcal{E} \to G^{p+q}\mathcal{E}$ for each $p, q \geq 1$. Applying the diagonalization functor, we get a pairing of simplicial sets

$$\mu : \mathrm{diag}G^p\mathcal{E} \wedge \mathrm{diag}G^q\mathcal{E} \to \mathrm{diag}G^{p+q}\mathcal{E}.$$

Corollary 4.2.4. *Suppose $\mathcal{E}$ is an exact category with a chosen 0 object, and $\otimes$ is a strictly associative biexact pairing on $\mathcal{E}$. Then the induced pairings $\mu : \mathrm{diag}G^p\mathcal{E} \wedge \mathrm{diag}G^q\mathcal{E} \to \mathrm{diag}G^{p+q}\mathcal{E}$ defined above for all $p, q \geq 1$ are associative, in the sense that the following diagram commutes for every $p, q, r \geq 1$.*

$$\begin{tikzcd}
\mathrm{diag}G^p\mathcal{E} \wedge \mathrm{diag}G^q\mathcal{E} \wedge \mathrm{diag}G^r\mathcal{E} \arrow[r, "\mu\wedge 1"] \arrow[d, "1\wedge\mu"'] & \mathrm{diag}G^{p+q}\mathcal{E} \wedge \mathrm{diag}G^r\mathcal{E} \arrow[d, "\mu"] \\
\mathrm{diag}G^p\mathcal{E} \wedge \mathrm{diag}G^{q+r}\mathcal{E} \arrow[r, "\mu"'] & \mathrm{diag}G^{p+q+r}\mathcal{E}
\end{tikzcd}$$

Suppose $\mathcal{F}$ is another exact category with a chosen 0 object with a biexact pairing $\otimes$, and suppose there is a functor $\varphi : \mathcal{E} \to \mathcal{F}$ preserving the pairings in the sense that the following diagram commutes (strictly, not up to a natural isomorphism).

$$\begin{tikzcd}
\mathcal{E} \times \mathcal{E} \arrow[r, "\otimes"] \arrow[d, "(\varphi,\varphi)"'] & \mathcal{E} \arrow[d, "\varphi"] \\
\mathcal{F} \times \mathcal{F} \arrow[r, "\otimes"'] & \mathcal{F}
\end{tikzcd}$$

Then the following diagram commutes for every $p, q \geq 1$.

$$\begin{tikzcd}
\mathrm{diag}G^p\mathcal{E} \wedge \mathrm{diag}G^q\mathcal{E} \arrow[r, "\mu"] \arrow[d, "\mathrm{diag}G^p\varphi\wedge\mathrm{diag}G^q\varphi"'] & \mathrm{diag}G^{p+q}\mathcal{E} \arrow[d, "\mathrm{diag}G^{p+q}\varphi"] \\
\mathrm{diag}G^p\mathcal{F} \wedge \mathrm{diag}G^q\mathcal{F} \arrow[r, "\mu"'] & \mathrm{diag}G^{p+q}\mathcal{F}
\end{tikzcd}$$

Proof. This follows immediately from Proposition 4.2.2 and Proposition 4.2.3. ∎

Chapter 5

The motivic K-theory spectrum $\mathcal{K}$

In this chapter, we define a motivic symmetric ring spectrum $\mathcal{K}$ representing algebraic K-theory. It is shown that $\mathcal{K}$ is equivalent to Voevodsky's motivic spectrum BGL when the base scheme is regular.

5.1 The construction

Let $\mathbb{P}^1_S$ be the projective line over a base scheme S considered as a pointed motivic space with the base point ∞. Recall that the motivic space represented by a scheme Z is the discrete simplicial presheaf $X \mapsto \mathrm{Hom}_{Sm/S}(X, Z)$. Let T be the mapping cylinder of the inclusion $i_\infty : S \to \mathbb{P}^1_S$ of the point at infinity. For each scheme X in Sm/S, $T(X)$ is the mapping cylinder of the inclusion $\mathrm{Hom}_{Sm/S}(X, S) \to \mathrm{Hom}_{Sm/S}(X, \mathbb{P}^1_S)$. The structure map $p_X : X \to S$ is the unique element of $\mathrm{Hom}_{Sm/S}(X, S)$ and its image in $\mathrm{Hom}_{Sm/S}(X, \mathbb{P}^1_S)$ is the composite map $X \xrightarrow{p_X} S \xrightarrow{i_\infty} \mathbb{P}^1_S$. Hence the mapping cylinder $T(X)$ is obtained by simply adding an edge connecting p_X and $i_\infty p_X$ to the discrete simplicial set $\mathbb{P}^1_S(X)$. The base point of $T(X)$ is chosen to be p_X. The motivic space T is homotopy equivalent to $\mathbb{P}^1_S$, and the motivic spectrum $\mathcal{K}$ constructed in this section will be a motivic T-spectrum. The reason we take T instead of $\mathbb{P}^1_S$ is that the structure maps need to preserve base points of motivic spaces. It will be explained more in section 5.1.2 when we define the structure maps of $\mathcal{K}$.

5.1.1 The space at the n-th level

For each $n \geq 1$, let $\mathcal{K}_n$ be the pointed motivic space

$$\mathcal{K}_n(X) = \mathrm{diag} G^n \mathbf{V}(X) \quad \text{for } X \in Sm/S \tag{5.1}$$

where G^n is the iterated G-construction and $\mathbf{V}(X)$ is the small exact category of standard vector bundles on X. (See Definition 4.1.4, Definition 3.3.1, and Theorem 3.3.10.) If $f : Y \to X$ is a morphism of Sm/S, it induces an exact functor $f^* : \mathbf{V}(X) \to \mathbf{V}(Y)$ by Theorem 3.3.10 (3) and the application of $\mathrm{diag} G^n$ gives a map of simplicial sets $f^* : \mathrm{diag}G^n\mathbf{V}(X) \to \mathrm{diag}G^n\mathbf{V}(Y)$. Again by Theorem 3.3.10 (3), if $g : Z \to Y$ is another morphism of Sm/S, then $(fg)^*$ and g^*f^* are the same map of simplicial sets: $\mathrm{diag}G^n\mathbf{V}(X) \to \mathrm{diag}G^n\mathbf{V}(Z)$, thus $\mathcal{K}_n$ is indeed a motivic space. The base point of $\mathcal{K}_n$ is the zero functor 0. When $n = 0$, we take $\mathcal{K}_0 = S_+$. Recall that for any motivic space M, M_+ is the pointed motivic space $M \coprod S$ based at S. With this definition, there are natural isomorphisms $\mathcal{K}_0 \wedge A \cong A \wedge \mathcal{K}_0 \cong A$ for any pointed motivic space A. The motivic T-spectrum $\mathcal{K}$ will have $\mathcal{K}_n$ as its n-th space.

5.1.2 The structure maps

For the construction of the structure maps $\sigma : T \wedge \mathcal{K}_n \to \mathcal{K}_{1+n}$, first we define a map $\eta : T \to \mathcal{K}_1$, which will be the first structure map $T \wedge \mathcal{K}_0 \to \mathcal{K}_1$. By definition, $\mathcal{K}_1 = G\mathbf{V}(-)$. First we define $\eta(X) : T(X) \to G\mathbf{V}(X)$ for each $X \in Sm/S$. Since $T(X)$ is a 1-dimensional simplicial set with exactly one edge, we need to specify the image of the vertices and the image of the edge. For $u \in \mathbb{P}^1_S(X) = \mathrm{Hom}_{Sm/S}(X, \mathbb{P}^1_S)$, let $\mathcal{L}_u = u^*\mathcal{O}_{\mathbb{P}^1_S}(-1)$ where $\mathcal{O}_{\mathbb{P}^1_S}(-1)$ is the twisted standard line bundle. (See section 3.3.3.) The map $\eta(X) : T(X) \to G\mathbf{V}(X)$ sends u to $(\mathcal{O}_X, \mathcal{L}_u)$. The edge of $T(X)$ has vertices p_X and $i_\infty p_X$. The vertex p_X is the base point, thus is sent to the base point (0,0) of $G\mathbf{V}(X)$. The other vertex $i_\infty p_X$ is sent to $(\mathcal{O}_X, \mathcal{O}_X)$, because

$$\mathcal{L}_{i_\infty p_X} = (i_\infty p_X)^*\mathcal{O}_{\mathbb{P}^1_S}(-1) = p_X^* i_\infty^* \mathcal{O}_{\mathbb{P}^1_S}(-1) = p_X^*\mathcal{O}_S = \mathcal{O}_X$$

by Theorem 3.3.10 (3), Lemma 3.3.11, and Theorem 3.3.8 (4). Then the edge of $T(X)$ has to be mapped to a 1-simplex of $G\mathbf{V}(X)$ with vertices $(0, 0)$ and $(\mathcal{O}_X, \mathcal{O}_X)$. The choice is $([\mathcal{O}_X], [\mathcal{O}_X])$ where $[\mathcal{O}_X]$ is the exact sequence $0 \rightarrowtail \mathcal{O}_X \overset{1}{\twoheadrightarrow} \mathcal{O}_X$.

Remark 5.1.1. Now the reader should be able to see why we needed to use the mapping cylinder T instead of $\mathbb{P}^1_S$. The structure map has been defined to model multiplication by $[\mathcal{O}] - [\mathcal{O}(-1)] \in K_0$ as in Voevodsky's spectrum BGL, but if we use $\mathbb{P}^1_S$, the problem is that the base point ∞ of $\mathbb{P}^1_S$ is mapped to $(\mathcal{O}_X, \mathcal{O}_X)$, which is not the base point of $G\mathbf{V}(X)$. Fortunately, there is a path between the base point $(0, 0)$ and $(\mathcal{O}_X, \mathcal{O}_X)$ in $G\mathbf{V}(X)$, and that is why we introduced the mapping cylinder T with a new base point.

If $f : Y \to X$ is a morphism of Sm/S, then the following diagram commutes,

$$\begin{tikzcd} T(X) \arrow[r, "\eta"] \arrow[d, "f^*"'] & \mathcal{K}_1(X) \arrow[d, "f^*"] \\ T(Y) \arrow[r, "\eta"'] & \mathcal{K}_1(Y) \end{tikzcd}$$

because for each $u : X \to \mathbb{P}^1_S$, we have $f^*\eta(u) = f^*(\mathcal{O}_X, u^*\mathcal{O}(-1)) = (f^*\mathcal{O}_X, f^*u^*\mathcal{O}(-1))$ and $\eta f^*(u) = \eta(uf) = (\mathcal{O}_Y, (uf)^*\mathcal{O}(-1))$, and they are the same by Theorem 3.3.8 (4) and Theorem 3.3.10 (3). Therefore, η is a map of motivic spaces.

For the construction of the structure maps in general, the next step is to define a pairing $\mu : \mathcal{K}_p \wedge \mathcal{K}_q \to \mathcal{K}_{p+q}$ for each $p, q, r \geq 0$ with certain properties. For each X, the tensor product operation on $\mathbf{V}(X)$ has been defined in section 3.3. From the discussion in section 4.2 applied to $\mathbf{V}(X)$, we obtain a pairing

$$\mu : \mathcal{K}_p(X) \wedge \mathcal{K}_q(X) \to \mathcal{K}_{p+q}(X)$$

This pairing is functorial in X by Corollary 4.2.4 and Theorem 3.3.8 (4). Therefore, we obtain a pairing of motivic spaces.

$$\mu : \mathcal{K}_p \wedge \mathcal{K}_q \to \mathcal{K}_{p+q}.$$

The tensor product operation on $\mathbf{V}(X)$ is strictly associative by Theorem 3.3.8 (1). (See Definition 4.2.1 for the meaning of strict associativity.) Therefore, by Corollary 4.2.4, we have the following commutative diagram for each $p, q, r \geq 0$, and each $X \in Sm/S$,

$$\begin{tikzcd}
\mathcal{K}_p(X) \wedge \mathcal{K}_q(X) \wedge \mathcal{K}_r(X) \arrow[r, "\mu\wedge 1"] \arrow[d, "1\wedge\mu"'] & \mathcal{K}_{p+q}(X) \wedge \mathcal{K}_r(X) \arrow[d, "\mu"] \\
\mathcal{K}_p(X) \wedge \mathcal{K}_{q+r}(X) \arrow[r, "\mu"'] & \mathcal{K}_{p+q+r}(X)
\end{tikzcd}$$

thus, we have the following commutative diagram of motivic spaces.

$$\begin{tikzcd}
\mathcal{K}_p \wedge \mathcal{K}_q \wedge \mathcal{K}_r \arrow[r, "\mu\wedge 1"] \arrow[d, "1\wedge\mu"'] & \mathcal{K}_{p+q} \wedge \mathcal{K}_r \arrow[d, "\mu"] \\
\mathcal{K}_p \wedge \mathcal{K}_{q+r} \arrow[r, "\mu"'] & \mathcal{K}_{p+q+r}
\end{tikzcd} \tag{5.2}$$

From this diagram, we deduce that the product maps are associative. We will denote the product map of multiple factors also by μ.

$$\mu : \mathcal{K}_{p_1} \wedge \mathcal{K}_{p_2} \wedge \cdots \wedge \mathcal{K}_{p_k} \to \mathcal{K}_{p_1+p_2+\cdots+p_k}$$

Define the structure map of $\mathcal{K}$ in general to be the composite map

$$\sigma : T \wedge \mathcal{K}_n \xrightarrow{\eta\wedge 1} \mathcal{K}_1 \wedge \mathcal{K}_n \xrightarrow{\mu} \mathcal{K}_{1+n}$$

for each $n \geq 0$. This defines $\mathcal{K}$ as a motivic spectrum. The associativity of μ leads to the following formulation of iterated structure maps $\sigma^p : T^p \wedge \mathcal{K}_n \to \mathcal{K}_{p+n}$. (Recall that T^p is the p-fold smash product of T, not the p-fold product.)

Lemma 5.1.2. *Let $\eta_p = \mu(\eta^p)$. Then $\sigma^p = \mu(\eta_p \wedge 1)$.*

$$\eta_p : T^p \xrightarrow{\eta^p} \mathcal{K}_1^p \xrightarrow{\mu} \mathcal{K}_p, \qquad \sigma^p : T^p \wedge \mathcal{K}_n \xrightarrow{\eta_p \wedge 1} \mathcal{K}_p \wedge \mathcal{K}_n \xrightarrow{\mu} \mathcal{K}_{p+n}$$

Proof. We prove it by induction on p. When $p = 1$, by definition, $\sigma = \mu(\eta \wedge 1)$. For $p > 1$, the following diagram commutes, where the top row is $1 \wedge \sigma$.

$$\begin{tikzcd}
T^{p-1} \wedge T \wedge \mathcal{K}_n \arrow[r, "1\wedge\eta\wedge 1"] \arrow[d, "\eta^{p-1}\wedge\eta\wedge 1"'] & T^{p-1} \wedge \mathcal{K}_1 \wedge \mathcal{K}_n \arrow[r, "1\wedge\mu"] \arrow[d, "\eta^{p-1}\wedge 1\wedge 1"'] & T^{p-1} \wedge \mathcal{K}_{1+n} \arrow[d, "\eta^{p-1}\wedge 1"] \\
\mathcal{K}_1^{p-1} \wedge \mathcal{K}_1 \wedge \mathcal{K}_n \arrow[r, "1"'] & \mathcal{K}_1^{p-1} \wedge \mathcal{K}_1 \wedge \mathcal{K}_n \arrow[r, "1\wedge\mu"'] & \mathcal{K}_1^{p-1} \wedge \mathcal{K}_{1+n}
\end{tikzcd}$$

By induction, the following diagram commutes.

$$\begin{tikzcd}
T^{p-1} \wedge \mathcal{K}_{1+n} \arrow[rr, "\sigma^{p-1}"] \arrow[d, "\eta^{p-1}\wedge 1"'] \arrow[drr, "\eta_{p-1}"] & & \mathcal{K}_{p+n} \\
\mathcal{K}_1^{p-1} \wedge \mathcal{K}_{1+n} \arrow[rr, "\mu"'] & & \mathcal{K}_{p-1} \wedge \mathcal{K}_{1+n} \arrow[u, "\mu"']
\end{tikzcd}$$

Connecting the diagrams together, we get that $\sigma^p = \sigma^{p-1}(1 \wedge \sigma)$ is the composite

$$T^{p-1} \wedge T \wedge \mathcal{K}_n \xrightarrow{\eta^p \wedge 1} \mathcal{K}_1^{p-1} \wedge \mathcal{K}_1 \wedge \mathcal{K}_n \xrightarrow{1\wedge\mu} \mathcal{K}_1^{p-1} \wedge \mathcal{K}_{1+n} \xrightarrow{\mu\wedge 1} \mathcal{K}_{p-1} \wedge \mathcal{K}_{1+n} \xrightarrow{\mu} \mathcal{K}_{p+n}$$

But the last three maps together is the same as $\mathcal{K}_1^p \wedge \mathcal{K}_n \xrightarrow{\mu\wedge 1} \mathcal{K}_p \wedge \mathcal{K}_n \xrightarrow{\mu} \mathcal{K}_{n+p}$ by the associativity of μ. Therefore, $\sigma^p = \mu(\eta_p \wedge 1)$.

$$\begin{tikzcd}
T^p \wedge \mathcal{K}_n \arrow[r, "\eta^p\wedge 1"] \arrow[dr, "\eta_p\wedge 1"'] & \mathcal{K}_1^p \wedge \mathcal{K}_n \arrow[d, "\mu\wedge 1"] & \\
& \mathcal{K}_p \wedge \mathcal{K}_n \arrow[r, "\mu"'] & \mathcal{K}_{p+n}
\end{tikzcd}$$

□

5.1.3 Symmetric group action

We will describe the action of symmetric groups on $\mathcal{K}$ and prove the equivariance of the iterated structure maps σ^p. Let $[n] = \{1, 2, \ldots, n\}$ be the set of n elements, which can be considered as the discrete category of n objects. Then each $\sigma \in \Sigma_n$ defines a functor $\sigma : [n] \to [n]$ permuting objects. Suppose $\mathcal{C}$ is a category. The product category $\mathcal{C}^n$ can be identified with the functor category whose objects are functors $[n] \to \mathcal{C}$ and whose morphisms are natural transformations of functors. For each $\sigma \in \Sigma_n$, we define a functor $\sigma^* : \mathcal{C}^n \to \mathcal{C}^n$ by right composition with σ. If we represent an object of $\mathcal{C}^n$ as an n-tuple of objects of $\mathcal{C}$, then σ^* is the functor

$$(C_1, C_2, \ldots, C_n) \mapsto (C_{\sigma(1)}, C_{\sigma(2)}, \ldots, C_{\sigma(n)}).$$

If $\sigma, \tau \in \mathcal{C}$, then $(\sigma\tau)^* = \tau^*\sigma^*$ by definition. Now suppose $\varphi : \mathcal{C} \to \mathcal{D}$ is a functor. Then it induces a functor $\varphi^n : \mathcal{C}^n \to \mathcal{D}^n$, and the following diagram commutes for any $\sigma \in \Sigma_n$.

$$\begin{tikzcd}
\mathcal{C}^n \arrow[r, "\sigma^*"] \arrow[d, "\varphi^n"'] & \mathcal{C}^n \arrow[d, "\varphi^n"] \\
\mathcal{D}^n \arrow[r, "\sigma^*"'] & \mathcal{D}^n
\end{tikzcd}$$

Suppose $X \in Sm/S$. Recall that an m-simplex of $\mathcal{K}_n(X)$ is a multiexact functor $\Gamma(A) \to \mathbf{V}(X)$ where $A = (\underline{m}, \ldots, \underline{m}) \in \Delta^n$, and $\Gamma(A)$ is the product category $\mathrm{Ar}(\gamma(\underline{m}))^n$. Then each $\sigma \in \Sigma_n$ induces a functor $\sigma^* : \Gamma(A) \to \Gamma(A)$ as above. For $\sigma \in \Sigma_n$ and $M \in \mathcal{K}_n(X)$, define σM to be the composite

$$\sigma M : \Gamma(A) \xrightarrow{\sigma^*} \Gamma(A) \xrightarrow{M} \mathbf{V}(X). \tag{5.3}$$

Then σM is again a multiexact functor since σ^* simply permutes coordinates. For every $\sigma, \tau \in \Sigma_n$, $(\sigma\tau)M = \sigma(\tau M)$, and $\sigma M = M$ if σ is the identity. So we have defined a left action of Σ_n on the set $\mathcal{K}_n(X)_m$ of m-simplices of $\mathcal{K}_n(X)$.

Suppose $\alpha : \underline{m'} \to \underline{m}$ is an order-preserving map. It induces a functor

$$\alpha_* : \mathrm{Ar}(\gamma(\underline{m'})) \to \mathrm{Ar}(\gamma(\underline{m})),$$

and the following diagram commutes for any $\sigma \in \Sigma_n$, where $A' = (\underline{m'}, \ldots, \underline{m'}) \in \Delta^n$.

$$\begin{tikzcd} \Gamma(A) \arrow[r, "\sigma^*"] \arrow[d, "\alpha^n_*"'] & \Gamma(A) \arrow[d, "\alpha^n_*"] \\ \Gamma(A') \arrow[r, "\sigma^*"'] & \Gamma(A') \end{tikzcd}$$

As a result, for any $\sigma \in \Sigma_n$ and $M \in \mathcal{K}_n(X)_m$,

$$\alpha^*\sigma M = M\sigma^*\alpha^n_* = M\alpha^n_*\sigma^* = \sigma\alpha^* M.$$

This shows that Σ_n acts on the simplicial set $\mathcal{K}_n(X)$.

One could understand how the symmetric group acts on $\mathcal{K}_n(X)$ by looking at a simple example. On the 0-skeletal level, a vertex in $\mathcal{K}_n(X) = \mathrm{diag} G^n \mathbf{V}(X)$ is an n-dimensional cube consisting of 2^n objects of $\mathbf{V}(X)$. Then $\sigma \in \Sigma_n$ sends it to the object obtained by permuting the coordinates of the cube. For example, if $n = 2$ and σ is the transposition in Σ_2, then $\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ is sent to $\begin{pmatrix} P & R \\ Q & S \end{pmatrix}$.

Suppose that $f : Y \to X$ is a map in Sm/S, and $\sigma \in \Sigma_n$. Then the diagram

$$\begin{tikzcd} \mathcal{K}_n(X) \arrow[r, "\sigma"] \arrow[d, "f^*"'] & \mathcal{K}_n(X) \arrow[d, "f^*"] \\ \mathcal{K}_n(Y) \arrow[r, "\sigma"'] & \mathcal{K}_n(Y) \end{tikzcd}$$

commutes because σ permutes the coordinates of both $\mathcal{K}_n(X)$ and $\mathcal{K}_n(Y)$ in the same way and f^* acts coordinate-wise. Finally, σ fixes the base point 0. This completes the description of the base point preserving left action of Σ_n on the motivic space $\mathcal{K}_n$.

Lemma 5.1.3. *The pairing $\mu : \mathcal{K}_p \wedge \mathcal{K}_q \to \mathcal{K}_{p+q}$ is $\Sigma_p \times \Sigma_q$-equivariant.*

Proof. Suppose $X \in Sm/S$, $\sigma \in \Sigma_p$, $\tau \in \Sigma_q$ and $\underline{m} \in \Delta$. Let $A = (\underline{m}, \ldots, \underline{m}) \in \Delta^p$ and $B = (\underline{m}, \ldots, \underline{m}) \in \Delta^q$. For multi-exact functors $U : \Gamma(A) \to \mathbf{V}(X)$ and $V : \Gamma(B) \to \mathbf{V}(X)$, their product $U \wedge V$ is the functor $\Gamma(A, B) \to \mathbf{V}(X)$ sending a multi-arrow $(\alpha, \beta) \in \Gamma(A) \times \Gamma(B)$ to $U\alpha \otimes V\beta$. The equation (5.3) tells us that $\sigma U \wedge \tau V$ is the functor $(\alpha, \beta) \mapsto U\sigma^*\alpha \otimes V\tau^*\beta$. On the other hand,

$(\sigma, \tau)(U \wedge V)$ is the functor

$$\begin{aligned}(\alpha, \beta) \mapsto (U \wedge V)(\sigma, \tau)^*(\alpha, \beta) &= (U \wedge V)(\sigma^*\alpha, \tau^*\beta) \\ &= U\sigma^*\alpha \otimes V\tau^*\beta.\end{aligned}$$

Therefore, $\sigma U \wedge \tau V = (\sigma, \tau)(U \wedge V)$. □

The strict commutativity of tensor product of standard vector bundles with standard line bundles is essential for the proof of the equivariance of structure maps of $\mathcal{K}$. The following lemma is the key to understand why.

Lemma 5.1.4. *The composite map*

$$\eta_2 : T \wedge T \xrightarrow{\eta \wedge \eta} \mathcal{K}_1 \wedge \mathcal{K}_1 \xrightarrow{\mu} \mathcal{K}_2$$

is Σ_2-equivariant.

Proof. Suppose $X \in Sm/S$. A simplex of $T(X) \wedge T(X)$ is represented by a pair of simplices of $T(X)$. We need to prove that for any pair (s, t) of simplices of $T(X)$,

$$\eta_2 \tau(s, t) = \tau \eta_2(s, t)$$

where $\tau = (1\ \ 2) \in \Sigma_2$ is a transposition. The left hand side is $\eta_2\tau(s, t) = \eta_2(t, s) = \mu(\eta t, \eta s)$. This is a biexact functor defined by

$$(\alpha, \beta) \mapsto ((\eta t)\alpha) \otimes ((\eta s)\beta), \qquad (f, g) \mapsto ((\eta t)f) \otimes ((\eta s)g)$$

on objects and morphisms, respectively. The right hand side is $\tau\eta_2(s, t) = \tau(\mu(\eta s, \eta t))$. This is a biexact functor defined by

$$(\alpha, \beta) \mapsto ((\eta s)\beta) \otimes ((\eta t)\alpha), \qquad (f, g) \mapsto ((\eta s)g) \otimes ((\eta t)f)$$

on objects and morphisms, respectively. Now observe that $(\eta s)\beta$ and $(\eta t)\alpha$ are standard line bundles or 0 by the way η is defined, and $(\eta s)g$ and $(\eta t)f$ are morphisms between them. Tensor product is strictly commutative on standard line bundles by Theorem 3.3.8 (3), and tensor product with 0 is 0 by Theorem 3.3.10 (4). Therefore, these two biexact functors are equal. □

Proposition 5.1.5. *The iterated structure map*

$$\sigma^p : T^p \wedge \mathcal{K}_q \to \mathcal{K}_{p+q}$$

is $\Sigma_p \times \Sigma_q$-equivariant for $p, q \geq 0$.

Proof. By Lemma 2.1.7, it suffices to prove the statement for $p \in \{1, 2\}$. We observed in Lemma 5.1.2 that $\sigma^p = \mu(\eta_p \wedge 1)$.

$$T^p \wedge \mathcal{K}_q \xrightarrow{\eta_p \wedge 1} \mathcal{K}_p \wedge \mathcal{K}_q \xrightarrow{\mu} \mathcal{K}_{p+q}$$

When $p = 1$, it is obvious that $\eta_1 \wedge 1$ is $\Sigma_1 \times \Sigma_q$-equivariant. When $p = 2$, $\eta_2 \wedge 1$ is $\Sigma_2 \times \Sigma_q$-equivariant by Lemma 5.1.4. The second map μ is $\Sigma_p \times \Sigma_q$-equivariant for all p by Lemma 5.1.3. Therefore, σ^p is $\Sigma_p \times \Sigma_q$-equivariant. □

The above proposition shows that the motivic spectrum $\mathcal{K}$ is a motivic *symmetric* spectrum.

5.1.4 Motivic symmetric ring spectrum $\mathcal{K}$

Now we use the pairings $\mu : \mathcal{K}_p \wedge \mathcal{K}_q \to \mathcal{K}_{p+q}$ to describe $\mathcal{K}$ as a motivic symmetric ring spectrum.

Proposition 5.1.6. *The motivic symmetric spectrum $\mathcal{K}$ is a motivic symmetric ring spectrum with the product map $\mu : \mathcal{K} \wedge \mathcal{K} \to \mathcal{K}$ determined by $\mu : \mathcal{K}_p \wedge \mathcal{K}_q \to \mathcal{K}_{p+q}$ for $p, q \geq 0$.*

Proof. First, we need to check the commutativity of the following diagrams in order to produce $\mu : \mathcal{K} \wedge \mathcal{K} \to \mathcal{K}$ from the component maps $\mu : \mathcal{K}_p \wedge \mathcal{K}_q \to \mathcal{K}_{p+q}$

$$\begin{tikzcd} T^r \wedge \mathcal{K}_p \wedge \mathcal{K}_q \arrow[r, "\sigma^r \wedge 1"] \arrow[d, "1 \wedge \mu"'] & \mathcal{K}_{r+p} \wedge \mathcal{K}_q \arrow[d, "\mu"] \\ T^r \wedge \mathcal{K}_{p+q} \arrow[r, "\sigma^r"'] & \mathcal{K}_{r+p+q} \end{tikzcd} \tag{5.4}$$

$$\begin{tikzcd} T^r \wedge \mathcal{K}_p \wedge \mathcal{K}_q \arrow[r, "t \wedge 1"] \arrow[d, "\sigma^r \wedge 1"'] & \mathcal{K}_p \wedge T^r \wedge \mathcal{K}_q \arrow[r, "1 \wedge \sigma^r"] & \mathcal{K}_p \wedge \mathcal{K}_{r+q} \arrow[d, "\mu"] \\ \mathcal{K}_{r+p} \wedge \mathcal{K}_q \arrow[r, "\mu"'] & \mathcal{K}_{r+p+q} \arrow[r, "\theta"'] & \mathcal{K}_{p+r+q} \end{tikzcd} \tag{5.5}$$

where $\theta \in \Sigma_{p+r+q}$ is the (r, p)-shuffle given by $\theta(i) = i + p$ for $1 \leq i \leq r$, $\theta(i) = i - r$ for $r + 1 \leq i \leq r + p$, and $\theta(i) = i$ for $r + p + 1 \leq i \leq r + p + q$.

The first diagram is commutative because it is factored as

$$\begin{tikzcd}
T^r \wedge \mathcal{K}_p \wedge \mathcal{K}_q \arrow[r, "\eta_r\wedge 1\wedge 1"] \arrow[d, "1\wedge\mu"'] & \mathcal{K}_r \wedge \mathcal{K}_p \wedge \mathcal{K}_q \arrow[r, "\mu\wedge 1"] \arrow[d, "1\wedge\mu"'] & \mathcal{K}_{r+p} \wedge \mathcal{K}_q \arrow[d, "\mu"] \\
T^r \wedge \mathcal{K}_{p+q} \arrow[r, "\eta_r\wedge 1"'] & \mathcal{K}_r \wedge \mathcal{K}_{p+q} \arrow[r, "\mu"'] & \mathcal{K}_{r+p+q}
\end{tikzcd}$$

and each part of the factorization is commutative. The second diagram is commutative essentially because of the commutativity of tensor product with standard line bundles. To demonstrate it, suppose $X \in Sm/S$, $\underline{m} \in \Delta^n$, $u_1, u_2, \ldots, u_r \in T(X)_m$, $P \in \mathcal{K}_p(X)_m$, and $Q \in \mathcal{K}_q(X)_m$. The image of $(u_1, \ldots, u_r)$ in $\mathcal{K}_r(X)_m$ via η_r is a multi-exact functor $U : \Gamma(\underline{m}, \ldots, \underline{m}) \to \mathbf{V}(X)$, whose value at each object of $\Gamma(\underline{m}, \ldots, \underline{m})$ is 0 or a tensor product of several standard line bundles. The images of $(u_1, \ldots, u_r, P, Q)$ along the maps $\mu(1 \wedge \sigma^r)(t \wedge 1)$ and $\theta\mu(\sigma^r \wedge 1)$ are, respectively, the multi-exact functors

$$(\alpha, \beta, \gamma) \mapsto P(\alpha) \otimes U(\beta) \otimes Q(\gamma),$$

$$(\alpha, \beta, \gamma) \mapsto U(\beta) \otimes P(\alpha) \otimes Q(\gamma),$$

for $\alpha \in \Gamma(A), \beta \in \Gamma(B), \gamma \in \Gamma(C)$ where $A = (\underline{m}, \ldots, \underline{m}) \in \Delta^p, B = (\underline{m}, \ldots, \underline{m}) \in \Delta^r$ and $C = (\underline{m}, \ldots, \underline{m}) \in \Delta^q$. Since $U(\beta)$ is 0 or of rank 1, $P(\alpha) \otimes U(\beta) \otimes Q(\gamma) = U(\beta) \otimes P(\alpha) \otimes Q(\gamma)$ by Theorem 3.3.8 (3). The same argument works for morphisms. Therefore, they are the same multi-exact functors.

Next, we describe the map from the unit. The unit of the symmetric monoidal category $\mathbf{SM}^{\Sigma}_{T}(S)$ is

$$\mathcal{T} = (T^0, T^1, T^2, \ldots, T^n, \ldots).$$

The unit map $\mathcal{T} \to \mathcal{K}$ is defined to be $\eta = (\eta_p : T^p \to \mathcal{K}_p)_{p \geq 0}$. Since the following diagram commutes, it is indeed a map of motivic symmetric spectra.

$$\begin{tikzcd}
T \wedge T^p \arrow[r, "1"] \arrow[d, "1\wedge\eta_p"'] & T^{1+p} \arrow[d, "\eta_{1+p}"] \\
T \wedge \mathcal{K}_p \arrow[r, "\sigma"'] & \mathcal{K}_{1+p}
\end{tikzcd}$$

The commutativity of the following diagrams of symmetric spectra

$$\begin{tikzcd} \mathcal{K}\wedge\mathcal{K}\wedge\mathcal{K} \arrow[r, "\mu\wedge 1"] \arrow[d, "1\wedge\mu"'] & \mathcal{K}\wedge\mathcal{K} \arrow[d, "\mu"] \\ \mathcal{K}\wedge\mathcal{K} \arrow[r, "\mu"] & \mathcal{K} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T}\wedge\mathcal{K} \arrow[r, "\eta\wedge 1"] \arrow[dr, "\sigma"'] & \mathcal{K}\wedge\mathcal{K} \arrow[d, "\mu"] & \mathcal{K}\wedge\mathcal{T} \arrow[l, "1\wedge\eta"'] \arrow[dl, "\rho"] \\ & \mathcal{K} & \end{tikzcd}$$

is deduced from the commutativity of the following diagrams for all p, q, and r.

$$\begin{tikzcd} \mathcal{K}_p\wedge\mathcal{K}_q\wedge\mathcal{K}_r \arrow[r, "\mu\wedge 1"] \arrow[d, "1\wedge\mu"'] & \mathcal{K}_{p+q}\wedge\mathcal{K}_r \arrow[d, "\mu"] \\ \mathcal{K}_p\wedge\mathcal{K}_{q+r} \arrow[r, "\mu"] & \mathcal{K}_{p+q+r} \end{tikzcd} \qquad \begin{tikzcd} T^p\wedge\mathcal{K}_q \arrow[r, "\eta_p\wedge 1"] \arrow[dr, "\sigma^p"'] & \mathcal{K}_p\wedge\mathcal{K}_q \arrow[d, "\mu"] & \mathcal{K}_p\wedge T^q \arrow[l, "1\wedge\eta_q"'] \arrow[dl, "\rho_q"] \\ & \mathcal{K}_{p+q} & \end{tikzcd}$$

The left diagram has been shown to be commutative in (5.2), the left-hand side of the right diagram is commutative by Lemma 5.1.2. The right unit map $\rho : \mathcal{K} \to \mathcal{T}$ is determined by $\rho_q : \mathcal{K}_p \wedge T^q \xrightarrow{t} T^q \wedge \mathcal{K}_p \xrightarrow{\sigma^q} \mathcal{K}_{q+p} \xrightarrow{c_{p,q}} \mathcal{K}_{p+q}$, and the right-hand side of the right diagram commutes by the same reason as the diagram (5.5) commutes, that is, by the strict commutativity of tensor product of standard vector bundles with standard line bundles, and the fact that the image of T^q in K_q consists of multi-exact functors whose values are tensor products of standard vector bundles of rank at most 1. $\square$

Remark 5.1.7. The motivic symmetric ring spectrum $\mathcal{K}$ is not commutative because the tensor product of standard vector bundles is not commutative in general. See Remark 3.3.9.

Suppose $f : S' \to S$ is a map of schemes. Recall from Proposition 2.4.1 that there is an adjunction $(f^*, f_*) : \mathbf{M}_\bullet(S) \to \mathbf{M}_\bullet(S')$ where $f_*(A)$ for $A \in \mathbf{M}_\bullet(S')$ is defined by $X \mapsto A(S' \times_S X)$.

Lemma 5.1.8. *Suppose $f : S' \to S$ is a map of schemes. Let T and T' be the mapping cylinders of $i_\infty : S \to \mathbb{P}^1_S$ and $i'_\infty : S' \to \mathbb{P}^1_{S'}$, respectively. Then there is an isomorphism $\epsilon : f^*T \to T'$.*

Proof. This is a variant of Yoneda lemma. We remark that for any $A \in \mathbf{M}_\bullet(S)$, a map $\varphi : T \to A$ of pointed motivic spaces is uniquely determined by a pair of a 0-simplex $u = \varphi_{\mathbb{P}^1_S}(1_{\mathbb{P}^1_S})$ of $A(\mathbb{P}^1_S)$ and a 1-simplex e of $A(\mathbb{P}^1_S)$ connecting the base point of A to $v = \varphi_{\mathbb{P}^1_S}(j_\infty)$ where j_∞ is the composite $\mathbb{P}^1_S \to S \xrightarrow{i_\infty} \mathbb{P}^1_S$. Let's call the set of all such pairs $A^\#(\mathbb{P}^1_S)$. Then

$$\mathrm{Hom}_{\mathbf{M}_\bullet(S)}(T, f_*A) \cong (f_*A)^\#(\mathbb{P}^1_S) \cong A^\#(S' \times_S \mathbb{P}^1_S) \cong A^\#(\mathbb{P}^1_{S'}) \cong \mathrm{Hom}_{\mathbf{M}_\bullet(S')}(T', A).$$

Therefore, T' is isomorphic to f^*T. $\square$

Theorem 5.1.9. *Suppose $f : S' \to S$ is a map of schemes. Let T' and $\mathcal{K}'$ denote the mapping cylinder of $i'_\infty : S' \to \mathbb{P}^1_{S'}$ and the K-theory spectrum over S' corresponding to T and $\mathcal{K}$ over S. Then there is a motivic symmetric T'-spectrum $f^*\mathcal{K}$ such that $(f^*\mathcal{K})_n = f^*(\mathcal{K}_n)$ and there is a map of motivic symmetric T'-spectrum $\varphi : f^*\mathcal{K} \to \mathcal{K}'$. If S' is smooth over S, then φ is an isomorphism. Moreover, if $\mathcal{K}$ is a motivic symmetric ring spectrum, then $f^*\mathcal{K}$ can be given the structure of a motivic symmetric ring spectrum such that φ respects monoidal structures.*

Proof. Let $f^{-1} : Sm/S \to Sm/S'$ denote the pullback functor $X \mapsto S' \times_S X$. For each $X \in Sm/S$, the projection map $\pi_X : f^{-1}X \to X$ induces an exact functor $\pi_X^* : \mathbf{V}(X) \to \mathbf{V}(f^{-1}X)$. Applying $\mathrm{diag}G^n$ to π_X^* gives us a map of simplicial sets

$$\mathcal{K}_n(X) \to \mathcal{K}'_n(f^{-1}X) = f_*\mathcal{K}'_n(X).$$

If $g : Y \to X$ is a map in Sm/S, then $f^{-1}g : f^{-1}Y \to f^{-1}X$ is a map in Sm/S', and the commutative diagram

$$\begin{tikzcd} f^{-1}Y \arrow[r, "f^{-1}g"] \arrow[d, "\pi_Y"'] & f^{-1}X \arrow[d, "\pi_X"] \\ Y \arrow[r, "g"'] & X \end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd} \mathbf{V}(X) \arrow[r, "g^*"] \arrow[d, "\pi_X^*"'] & \mathbf{V}(Y) \arrow[d, "\pi_Y^*"] \\ \mathbf{V}(f^{-1}X) \arrow[r, "(f^{-1}g)^*"'] & \mathbf{V}(f^{-1}Y) \end{tikzcd}$$

by Theorem 3.3.10 (3). After applying $\mathrm{diag}G^n$, we get the following commutative diagram.

$$\begin{tikzcd} \mathcal{K}_n(X) \arrow[r, "g^*"] \arrow[d, "\pi_X^*"'] & \mathcal{K}_n(Y) \arrow[d, "\pi_Y^*"] \\ f_*\mathcal{K}'_n(X) \arrow[r, "(f^{-1}g)^*"'] & f_*\mathcal{K}'_n(Y) \end{tikzcd}$$

Thus, we have defined a map of motivic spaces $\mathcal{K}_n \to f_*\mathcal{K}'_n$. This map is Σ_n-equivariant and base point preserving. By Proposition 2.4.1, we obtain a Σ_n-equivariant map of pointed motivic spaces $f^*\mathcal{K}_n \to \mathcal{K}'_n$. To prove that these maps define a map of motivic symmetric f^*T-spectra

$$\varphi : f^*\mathcal{K} \to \mathcal{K}',$$

we need to check the commutativity of the following diagram.

$$\begin{tikzcd}
f^*T \wedge f^*\mathcal{K}_n \arrow[r, "\sigma_1"] \arrow[d, "1\wedge\varphi_n"'] & f^*\mathcal{K}_{n+1} \arrow[d, "\varphi_{1+n}"] \\
f^*T \wedge \mathcal{K}'_n \arrow[r, "\sigma_2"'] & \mathcal{K}'_{1+n}
\end{tikzcd}$$

Suppose $X' \in Sm/S'$. An m-simplex of $f^*T(X')$ is represented by (X, h, u) where $X \in Sm/S$, $h : X' \to f^{-1}X$, and $u \in T(X)_m$. Similarly, an m-simplex of $f^*\mathcal{K}_n(X')$ is represented by (X_1, h_1, M) where $X_1 \in Sm/S$, $h_1 : X' \to f^{-1}X_1$, and $M \in \mathcal{K}_n(X_1)_m$. We may assume that $X = X_1$ and $h = h_1$ since we can replace them by $X_2 = X \times_S X_1$ and $h_2 : X' \to f^{-1}X_2$, and replace u and M by their pullbacks. Elaborating σ_1 and σ_2, we get the following.

$$\varphi_{1+n}\sigma_1 : f^*T \wedge f^*\mathcal{K}_n \xrightarrow{\cong} f^*(T \wedge \mathcal{K}_n) \xrightarrow{f^*(\eta\wedge 1)} f^*(\mathcal{K}_1 \wedge \mathcal{K}_n) \xrightarrow{f^*\mu} f^*(\mathcal{K}_{1+n}) \xrightarrow{\varphi_{1+n}} \mathcal{K}'_{1+n} \tag{5.6}$$

$$\sigma_2(1 \wedge \varphi_n) : f^*T \wedge f^*\mathcal{K}'_n \xrightarrow{1\wedge\varphi_n} f^*T \wedge \mathcal{K}'_n \xrightarrow{\epsilon\wedge 1} T' \wedge \mathcal{K}'_n \xrightarrow{\eta'\wedge 1} \mathcal{K}'_1 \wedge \mathcal{K}'_n \xrightarrow{\mu'} \mathcal{K}'_{1+n} \tag{5.7}$$

The map $\varphi_n : f^*\mathcal{K}_n(X') \to \mathcal{K}'_n(X')$ is, by definition, $(X, h, M) \mapsto (\pi_X h)^*M$. Hence, an m-simplex $((X, h, u), (X, h, M))$ of $(f^*T \wedge f^*\mathcal{K}_n)(X')$ is sent via (5.6) and (5.7) to $(\pi_X h)^*\mu(\eta u, M)$ and $\mu'(\eta'\epsilon(X, m, U), (\pi_X h)^*M)$, respectively. Their values at $(\alpha, \beta) \in \Gamma(\underline{m}) \times \Gamma(\underline{m}, \ldots, \underline{m})$ are

$$(\pi_X h)^*\mu(\eta u, M)(\alpha, \beta) = (\pi_X h)^*(\eta u\alpha \otimes M\beta) = (\pi_X h)^*\eta u\alpha \otimes (\pi_X h)^*M\beta,$$

$$\mu'(\eta'\epsilon(X, m, u), (\pi_X h)^*M)(\alpha, \beta) = \eta'\epsilon(X, m, u)\alpha \otimes (\pi_X h)^*M\beta.$$

From these, we see that it is enough to prove that $(\pi_X h)^*\eta u = \eta'\epsilon(X, m, u)$, i.e., the commutativity of the following diagram,

$$\begin{tikzcd}
f^*T \arrow[r, "f^*\eta"] \arrow[d, "\epsilon"'] & f^*\mathcal{K}_1 \arrow[d, "\varphi_1"] \\
T' \arrow[r, "\eta'"'] & \mathcal{K}'_1
\end{tikzcd}$$

or the commutativity of the adjoint diagram.

$$\begin{tikzcd}
T \arrow[r, "\eta"] \arrow[d] & \mathcal{K}_1 \arrow[d] \\
f_*T' \arrow[r, "f_*\eta'"'] & f_*\mathcal{K}'_1
\end{tikzcd}$$

For any $X \in Sm/S$ and $u : X \to \mathbb{P}^1_S$, the images of u in $f_*\mathcal{K}'_1(X) = \mathcal{K}'_1(f^{-1}X) = G\mathbf{V}(f^{-1}X)$ along the two different paths are $(\pi_X^*\mathcal{O}_X, \pi_X^* u^*\mathcal{O}_{\mathbb{P}^1_S}(-1))$ and $(\mathcal{O}_{f^{-1}X}, (f^{-1}u)^*\mathcal{O}_{\mathbb{P}^1_{S'}}(-1))$. But $\pi_X^*\mathcal{O}_X = \mathcal{O}_{f^{-1}X}$ by Theorem 3.3.8 (4) and $\pi_X^* u^*\mathcal{O}_{\mathbb{P}^1_S}(-1) = (f^{-1}u)^*\mathcal{O}_{\mathbb{P}^1_{S'}}(-1)$ by Theorem 3.3.10 (3) and Lemma 3.3.12. Similarly, the edge of $T(X)$ is mapped to the same simplex of $f_*\mathcal{K}'_1(X)$ along the two paths. This completes the proof that $\varphi : f^*\mathcal{K} \to \mathcal{K}'$ is a map of motivic symmetric f^*T-spectra. By Lemma 5.1.8, we can identify f^*T with T', and the map is is a map of T'-spectra. Now suppose $f : S' \to S$ is smooth. Then for any $X' \in Sm/S'$, X' is considered as a smooth scheme over S as well. Then $f^*\mathcal{K}_n(X') \cong \mathcal{K}_n(X')$ and the map $\varphi_n : f^*\mathcal{K}_n(X') \cong \mathcal{K}(X') \to \mathcal{K}'(X')$ is the map $f^*\mathcal{K}_n(X') \cong \mathrm{diag}G^n\mathbf{V}(X') \to \mathrm{diag}G^n\mathbf{V}(X')$ induced by the identity map $X' \to X'$. Therefore, $\varphi : f^*\mathcal{K} \to \mathcal{K}'$ is an isomorphism. If $\mathcal{K}$ is a symmetric ring spectrum, then the product maps $\mu : \mathcal{K}_p \wedge \mathcal{K}_q \to \mathcal{K}_{p+q}$ induce the product maps for $f^*\mathcal{K}$.

$$f^*\mathcal{K}_p \wedge f^*\mathcal{K}_q \xrightarrow{\cong} f^*(\mathcal{K}_p \wedge \mathcal{K}_q) \xrightarrow{f^*\mu} f^*\mathcal{K}_{p+q}$$

The unit map of $f^*\mathcal{K}$ is induced by the unit map η of $\mathcal{K}$.

$$(T')^p \xrightarrow{\cong} f^*(T^p) \xrightarrow{f^*\eta_p} f^*\mathcal{K}_p$$

Therefore, $f^*\mathcal{K}$ is a symmetric ring spectrum. The map φ respects monoidal structures because $f^*\mathcal{K}_p \wedge f^*\mathcal{K}_q \xrightarrow{\varphi_p \wedge \varphi_q} \mathcal{K}'_p \wedge \mathcal{K}'_q$ is induced by the map $\mathcal{K}_p \wedge \mathcal{K}_q \xrightarrow{\pi_p \wedge \pi_q} f_*(\mathcal{K}_p) \wedge f_*(\mathcal{K}_q) = f_*(\mathcal{K}_p \wedge \mathcal{K}_q)$, and the following diagram commutes.

$$\begin{tikzcd}
f^*\mathcal{K}_p \wedge f^*\mathcal{K}_q \arrow[r, "\cong"] \arrow[d, "\varphi_p \wedge \varphi_q"] & f^*(\mathcal{K}_p \wedge \mathcal{K}_q) \arrow[r] & f^*\mathcal{K}_{p+q} \arrow[d, "\varphi_{p+q}"'] \\
\mathcal{K}'_p \wedge \mathcal{K}'_q \arrow[rr] & & \mathcal{K}'_{p+q}
\end{tikzcd}$$

□

5.2 Equivalence of $\mathcal{K}$ with Voevodsky's BGL

We first review the definition of Voevodsky's motivic spectrum BGL representing algebraic K-theory from [24, 17]. We assume that the base scheme S is regular throughout this section unless otherwise indicated. Let $Gr = \varinjlim Gr(n, 2n)$ be the infinite Grassmannian. The n-th space of the spectrum BGL is $BGL_n = Ex^{\mathbb{A}^1}(\mathbb{Z} \times Gr)$ for every $n \geq 0$. In order to define the structure map $\mathbb{P}^1_S \wedge BGL_n \to BGL_{n+1}$, he proves that there is an isomorphism

$$\mathrm{Hom}_{\mathbf{H}_\bullet(S)}(\mathbb{P}^1_S \wedge (\mathbb{Z}\times Gr), \mathbb{Z}\times Gr) \cong \mathrm{Hom}_{\mathbf{H}_\bullet(S)}(\mathbb{Z}\times Gr, \mathbb{Z}\times Gr).$$

The lifting of the map corresponding to the identity map of $\mathbb{Z}\times Gr$ is defined to be the structure map of BGL at every level. He makes use of the following three theorems and the projective bundle theorem of K-theory (Theorem 2.1 in [21]) to prove this bijection.

Theorem 5.2.1 (6.5 [24], 3.3.13 [17]). *For any $X \in Sm/S$ and any $i \geq 0$, there is a canonical map*

$$K_i^{TT}(X) \to \mathrm{Hom}_{\mathbf{H}_\bullet(S)}(S^i \wedge X_+, \mathbb{Z}\times Gr)$$

which is bijective if S is regular. (The groups on the left $K_i^{TT}(X)$ are the Thomason Troubaugh K-groups.)

Corollary 5.2.2 (6.6 [24]). *Let (X,x) and (Y,y) be schemes in Sm/S with basepoints. Then for any $i \geq 0$ there are canonical maps*

$$\begin{aligned} K_i(X,x) &\to \mathrm{Hom}_{\mathbf{H}_\bullet(S)}(S^i \wedge (X,x), \mathbb{Z}\times Gr) \\ K_i((X,x)\wedge(Y,y)) &\to \mathrm{Hom}_{\mathbf{H}_\bullet(S)}(S^i \wedge (X,x)\wedge(Y,y), \mathbb{Z}\times Gr) \end{aligned}$$

which are bijections if S is regular.

Lemma 5.2.3 (6.7[24]). *Let $(X_n, i_n : (X_n, x_n) \to (X_{n+1}, x_{n+1}))$ be an inductive system of pointed spaces such that all the morphisms i_n are monomorphisms and (Y,y) be a pointed space such that all the maps*

$$\mathrm{Hom}_{\mathbf{H}_\bullet(S)}(S^1 \wedge (X_{n+1}, x_{n+1}), (Y,y)) \to \mathrm{Hom}_{\mathbf{H}_\bullet(S)}(S^1 \wedge (X_n, x_n), (Y,y))$$

induced by $1 \wedge i_n$ are surjective. Then the canonical map

$$\mathrm{Hom}_{\mathbf{H}_\bullet(S)}\left(\varinjlim_n (X_n, x_n), (Y,y)\right) \to \varprojlim_n \mathrm{Hom}_{\mathbf{H}_\bullet(S)}((X_n, x_n), (Y,y))$$

is bijective.

In [24] Voevodsky argues that the embeddings of Grassmannians $G(d,2d) \to G(d+1, 2d+2)$ induce surjections of K-groups $K_n(Gr(d+1,2d+2)) \to K_n(Gr(d,2d))$ so that he can apply the above lemma. The projective bundle theorem implies that there is an isomorphism $K_n(X,x) \to K_n(\mathbb{P}^1_S \wedge (X,x))$ defined by multiplication by $[\mathcal{O}] - [\mathcal{O}(-1)] \in K_0(\mathbb{P}^1_S)$. Then Voevodsky gets a chain of bijections

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{H}_\bullet(S)}(\mathbb{P}^1_S \wedge (\mathbb{Z} \times Gr), (\mathbb{Z} \times Gr)) &\cong \varprojlim_d \mathrm{Hom}_{\mathbf{H}_\bullet(S)}\left(\mathbb{P}^1_S \wedge \coprod_{i=-d}^{d} Gr(d, 2d), \mathbb{Z} \times Gr\right) \\
&\cong \varprojlim_d K_0\left(\mathbb{P}^1_S \wedge \coprod_{i=-d}^{d} Gr(d, 2d)\right) \\
&\cong \varprojlim_d K_0\left(\coprod_{i=-d}^{d} Gr(d, 2d)\right) \\
&\cong \varprojlim_d \mathrm{Hom}_{\mathbf{H}_\bullet(S)}\left(\coprod_{i=-d}^{d} Gr(d, 2d), \mathbb{Z} \times Gr\right) \\
&\cong \mathrm{Hom}_{\mathbf{H}_\bullet(S)}(\mathbb{Z} \times Gr, \mathbb{Z} \times Gr)
\end{aligned}$$

We can prove theorems analogous to 5.2.1 and 5.2.2 for $\mathcal{K}$ in place of $\mathbb{Z} \times Gr$, then prove the equivalence of $\mathcal{K}$ and BGL. The next theorem is analogous to Proposition 3.3.9 in [17].

Theorem 5.2.4. *Suppose $X \in Sm/S$. There is a canonical isomorphism*

$$\mathrm{Hom}_{H^s(\mathbf{M}_\bullet(S))}(S^i \wedge X_+, \mathcal{K}_n) \cong K_i^{TT}(X)$$

for all $i \geq 0$ and $n \geq 1$ where K^{TT} is Thomason-Trobaugh K-theory [23].

Proof. By Lemma 4.1.5, it is enough to prove the isomorphism for $n = 1$. By Theorem 4.1.2, $\mathcal{K}_1$ induces the presheaf K of Quillen K-theory spectra $X \mapsto K(X)$. There is a natural map $K \to K^{TT}$ which is a simplicial weak equivalence by [23, 3.9]. Since [23, 10.8] and [16, 3.20] (cf. [12, 3.3]) implies that K^{TT} is simplicially fibrant, the theorem now follows from the isomorphism of Lemma 2.2.2:

$$\mathrm{Hom}_{H^s(\mathbf{M}_\bullet(S))}(S^i \wedge X_+, \mathcal{K}_1) \cong \pi_i((Ex^s\mathcal{K}_1)(X)) \cong \pi_i(K^{TT}(X)).$$

□

Theorem 5.2.5. *If S is regular, and X is in Sm/S, then there is a canonical isomorphism*

$$\mathrm{Hom}_{\mathbf{H}_\bullet(S)}(S^i \wedge X_+, \mathcal{K}_n) \cong K_i^{TT}(X)$$

for all $i \geq 0$ and $n \geq 1$.

Proof. This follows from Theorem 5.2.4, Proposition 2.2.1 and homotopy invariance of algebraic K-theory over regular schemes [23, 6.8] (cf. [17, 3.3.13]). ☐

Corollary 5.2.6. *Let (X,x) and (Y,y) be pointed smooth schemes over S. There are isomorphisms*

$$\mathrm{Hom}_{\mathbf{H}_\bullet(S)}(S^i \wedge (X,x), \mathcal{K}_n) \cong K_i(X,x) \tag{5.8}$$

$$\mathrm{Hom}_{\mathbf{H}_\bullet(S)}(S^i \wedge X_+ \wedge (Y,y), \mathcal{K}_n) \cong K_i(X \times Y, X \times y) \tag{5.9}$$

$$\mathrm{Hom}_{\mathbf{H}_\bullet(S)}(S^i \wedge (X,x) \wedge (Y,y), \mathcal{K}_n) \cong K_i((X,x) \wedge (Y,y)) \tag{5.10}$$

for all $i \geq 0$ and $n \geq 1$.

Proof. The inclusion $\{x\} \to X$ induces isomorphisms of pointed motivic spaces $(S^i \wedge X_+)/(S^i \wedge x_+) \cong S^i \wedge (X_+/x_+) \cong S^i \wedge (X,x)$. Thus by completing the top row of the following commutative diagram with split exact columns, we obtain the isomorphism (5.8).

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathrm{Hom}_{\mathbf{H}_\bullet(S)}(S^i \wedge (X,x), \mathcal{K}_n) \arrow[d] & K_i(X,x) \arrow[d] \\
\mathrm{Hom}_{\mathbf{H}_\bullet(S)}(S^i \wedge X_+, \mathcal{K}_n) \arrow[r, "\cong"] \arrow[d] & K_i(X) \arrow[d] \\
\mathrm{Hom}_{\mathbf{H}_\bullet(S)}(S^i \wedge x_+, \mathcal{K}_n) \arrow[r, "\cong"] \arrow[d] & K_i(x) \arrow[d] \\
0 & 0
\end{tikzcd}$$

The other isomorphisms are derived similarly considering the inclusions $x \times Y \to X \times Y$ and $X \times y \to X \times Y$. ☐

Proposition 5.2.7. *The map $\mathcal{K}_n \to \Omega_T \mathcal{K}_{n+1}$ induced by the structure map $\sigma : T \wedge \mathcal{K}_n \to \mathcal{K}_{n+1}$ is a motivic weak equivalence for all $n \geq 1$, (but not for $n = 0$).*

Proof. We will use an alternative formulation of motivic weak equivalence from [24]. By Definition 3.4, Theorem 3.6, and Lemma 3.8 of [24], to show that $\mathcal{K}_n \to \Omega_T \mathcal{K}_{n+1}$ is a motivic weak equivalence is to show that the induced map of the motivic homotopy groups

$$\pi_i^{\mathbb{A}^1}(\mathcal{K}_n)(X) = \mathrm{Hom}_{\mathbf{H}_\bullet(S)}(S^i \wedge X_+, \mathcal{K}_n) \to \mathrm{Hom}_{\mathbf{H}_\bullet(S)}(S^i \wedge X_+, \Omega_T \mathcal{K}_{n+1}) = \pi_i^{\mathbb{A}^1}(\Omega_T \mathcal{K}_{n+1})(X)$$

is an isomorphism for all $i \geq 0$ and all $X \in Sm/S$. By adjointness, we need to prove that the composite induced by the structure map

$$\begin{aligned}\mathrm{Hom}_{\mathbf{H}_\bullet(S)}(S^i \wedge X_+, \mathcal{K}_n) &\to \mathrm{Hom}_{\mathbf{H}_\bullet(S)}(T \wedge S^i \wedge X_+, T \wedge \mathcal{K}_n)\\ &\to \mathrm{Hom}_{\mathbf{H}_\bullet(S)}(T \wedge S^i \wedge X_+, \mathcal{K}_{n+1})\end{aligned}$$

is an isomorphism. By Corollary 5.2.6 and the weak equivalence $T \to \mathbb{P}^1_S$, we can identify it with the map

$$K_i(X) \to K_i(\mathbb{P}^1_S \times X, \infty \times X).$$

From the construction of the structure map σ, we see that this map is the multiplication map by the class $[\mathcal{O}] - [\mathcal{O}(-1)]$ in $K_0(\mathbb{P}^1_S)$, and by the projective bundle theorem of K-theory, it is an isomorphism. $\square$

Theorem 5.2.8. *There is an isomorphism $w_n : \mathbb{Z} \times Gr \to \mathcal{K}_n$ in $\mathbf{H}_\bullet(S)$ for $n \geq 1$.*

Proof. This theorem is essentially due to Morel and Voevodsky [17, 4.3.10]. In the simplicial homotopy category $H^s(\mathbf{M}_\bullet(S))$, the simplicial sheaf $(\mathbf{R}\Omega^1_s)\mathrm{B}(\coprod_{n\geq 0} \mathrm{BGL}_n)$ of Proposition 4.3.9 of [17] and the motivic space $\mathcal{K}_n$ are isomorphic since both represent the loop space of Quillen K-theory. By Proposition 4.3.10 of [17], there is a motivic weak equivalence

$$\mathbb{Z} \times Gr \to (\mathbf{R}\Omega^1_s)\mathrm{B}(\coprod_{n\geq 0} \mathrm{BGL}_n).$$

Hence the theorem follows. $\square$

This theorem and its proof shows in particular that the following diagram commutes for any $X \in Sm/S$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{H}_\bullet(S)}(X_+, \mathbb{Z} \times Gr) \arrow[d, "(w_n)_*"'] & \\
& K_0(X) \arrow[ul, "\cong"'] \arrow[dl, "\cong"] \\
\mathrm{Hom}_{\mathbf{H}_\bullet(S)}(X_+, \mathcal{K}_n) &
\end{tikzcd}$$

where maps from $K_0(X)$ are isomorphisms of Theorem 5.2.1 and 5.2.5, and $(w_n)_*$ is the composition with w_n. Similar diagrams for (X, x) and $(X, x) \wedge (Y, y)$ also commute by Corollary 5.2.2 and 5.2.6.

Theorem 5.2.9. *If the base scheme S is regular, then there is a motivic spectrum C and maps of motivic* *spectra $\mathcal{K} \xleftarrow{\varphi} C \xrightarrow{\psi} BGL$ such that ψ is a level equivalence after the first term, and φ is a level equivalence. In particular, $\mathcal{K}$ and BGL are stably equivalent as motivic spectra.*

Proof. Consider the following diagram in $\mathbf{H}_\bullet(S)$.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{H}_\bullet(S)}\left(\coprod_{i=-d}^{d} Gr(d,2d), \mathbb{Z}\times Gr\right) \arrow[r] \arrow[dd, bend right=60, "(w_n)_*"'] & \mathrm{Hom}_{\mathbf{H}_\bullet(S)}\left(\mathbb{P}^1_S \wedge \coprod_{i=-d}^{d} Gr(d,2d), \mathbb{Z}\times Gr\right) \arrow[dd, bend left=60, "(w_{n+1})_*"] \\
K_0\left(\coprod_{i=-d}^{d} Gr(d,2d)\right) \arrow[r] \arrow[u, "\cong"'] \arrow[d, "\cong"] & K_0\left(\mathbb{P}^1_S \wedge \coprod_{i=-d}^{d} Gr(d,2d)\right) \arrow[u, "\cong"] \arrow[d, "\cong"'] \\
\mathrm{Hom}_{\mathbf{H}_\bullet(S)}\left(\coprod_{i=-d}^{d} Gr(d,2d), \mathcal{K}_n\right) \arrow[r] & \mathrm{Hom}_{\mathbf{H}_\bullet(S)}\left(\mathbb{P}^1_S \wedge \coprod_{i=-d}^{d} Gr(d,2d), \mathcal{K}_{n+1}\right)
\end{tikzcd}$$

The middle row is the map of multiplication by $[\mathcal{O}] - [\mathcal{O}(-1)]$, which is an isomorphism by the projective bundle theorem. The top row is the composite of three isomorphisms just as in the discussion before Theorem 5.2.4. Then the upper part of the diagram commutes by definition. The bottom row is the composite of the following maps

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{H}_\bullet(S)}\left(\coprod_{i=-d}^{d} Gr(d,2d), \mathcal{K}_n\right) &\to \mathrm{Hom}_{\mathbf{H}_\bullet(S)}\left(\mathbb{P}^1_S \wedge \coprod_{i=-d}^{d} Gr(d,2d), \mathbb{P}^1_S \wedge \mathcal{K}_n\right) \\
&\to \mathrm{Hom}_{\mathbf{H}_\bullet(S)}\left(\mathbb{P}^1_S \wedge \coprod_{i=-d}^{d} Gr(d,2d), T \wedge \mathcal{K}_n\right) \\
&\to \mathrm{Hom}_{\mathbf{H}_\bullet(S)}\left(\mathbb{P}^1_S \wedge \coprod_{i=-d}^{d} Gr(d,2d), \mathcal{K}_{n+1}\right)
\end{aligned}$$

which is induced by the structure map $\sigma : T \wedge \mathcal{K}_n \to \mathcal{K}_{n+1}$ of the spectrum $\mathcal{K}$. Then the lower part of the diagram commutes by the construction of σ. After applying $\varprojlim_d$ to the diagram, we get the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{H}_\bullet(S)}(\mathbb{Z}\times Gr, \mathbb{Z}\times Gr) \arrow[r] \arrow[d, "(w_n)_*"'] & \mathrm{Hom}_{\mathbf{H}_\bullet(S)}(\mathbb{P}^1_S \wedge (\mathbb{Z}\times Gr), \mathbb{Z}\times Gr) \arrow[d, "(w_{n+1})_*"] \\
\mathrm{Hom}_{\mathbf{H}_\bullet(S)}(\mathbb{Z}\times Gr, \mathcal{K}_n) \arrow[r] & \mathrm{Hom}_{\mathbf{H}_\bullet(S)}(\mathbb{P}^1_S \wedge (\mathbb{Z}\times Gr), \mathcal{K}_{n+1})
\end{tikzcd}$$

Following two different paths from upper left corner to lower right corner, the identity map of $\mathbb{Z}\times Gr$ is sent to $w_{n+1}\varepsilon$ and $(\rho^{-1} \wedge \sigma)w_n$ where ε is the map $\mathbb{P}^1_S \wedge (\mathbb{Z}\times Gr) \to \mathbb{Z}\times Gr$ that defines the structure map of BGL when lifted, and $\rho : T \to \mathbb{P}^1_S$ is the deformation retract, which is an isomorphism in the homotopy category. This implies that the following diagram commutes in the homotopy category $\mathbf{H}_\bullet(S)$.

$$\begin{tikzcd} T \wedge \mathcal{K}_n \arrow[r, "\sigma"] \arrow[d, "\rho \wedge w_n^{-1}"'] & \mathcal{K}_{n+1} \arrow[d, "w_{n+1}^{-1}"] \\ \mathbb{P}^1_S \wedge (\mathbb{Z} \times Gr) \arrow[r, "\varepsilon"'] & \mathbb{Z} \times Gr \end{tikzcd}$$

By Proposition 2.2.4, the motivic spectra $\mathcal{K}$ and BGL are stably equivalent. (The proposition implies that we can construct zig-zag equivalences $\mathcal{K} \xleftarrow{\sim} C \xrightarrow{\sim} BGL$ after the first term. We could take the first term of C to be S_+.) $\square$

A monoidal structure of BGL is discussed by Panin, Pimenov, and Röndigs following Voevodsky in [20]. We first show that the monoidal structure of $\mathcal{K}$ induces a monoidal structure of BGL, then review their definition of monoidal structure of BGL. Then we finally show that the two structures are compatible.

Let $U : \mathbf{SM}^\Sigma(S) \to \mathbf{SM}(S)$ be the forgetful functor. It is right adjoint to the symmetrization functor $V : \mathbf{SM}(S) \to \mathbf{SM}^\Sigma(S)$. The adjoint pair (V, U) is a Quillen equivalence. They induce equivalences of stable homotopy categories $(LV, RU) : \mathbf{SH}(S) \rightleftarrows \mathbf{SH}^\Sigma(S)$. (See Theorem 2.2.3.) The right derived functor RU is the composition of U and the stably fibrant replacement functor $A \mapsto A^{sf}$ of motivic symmetric spectra. The symmetric monoidal model structure of $\mathbf{SM}^\Sigma(S)$ induces the symmetric monoidal structure of $\mathbf{SH}^\Sigma(S)$ by Theorem 2.3.11, and then the symmetric monodial structure of $\mathbf{SH}(S)$ is obtained by these equivalences. In particular, the product of motivic spectra A and B is defined by

$$A \wedge B = RU(LVA \wedge LVB).$$

Since the K-theory spectrum $\mathcal{K}$ is a monoid in $\mathbf{SM}^\Sigma(S)$, it becomes a monoind in $\mathbf{SH}^\Sigma(S)$ by Lemma 2.3.7. Then $RU\mathcal{K}$ is a monoid in $\mathbf{SH}(S)$. The product map of $RU\mathcal{K}$ is defined by

$$\mu_{RU\mathcal{K}} : RU\mathcal{K} \wedge RU\mathcal{K} = RU(LVRU\mathcal{K} \wedge LVRU\mathcal{K}\wedge) \cong RU(\mathcal{K} \wedge \mathcal{K}) \xrightarrow{RU\mu} RU\mathcal{K} \tag{5.11}$$

where the isomorphism in the middle is induced by the natural isomorphism $LVRU \cong 1_{\mathbf{SH}^\Sigma(S)}$. Theorem 5.2.9 shows that $U\mathcal{K}$ is stably equivalent to BGL. We will also prove that $RU\mathcal{K}$ is stably equivalent to BGL, so that the multiplicative structure of $\mathcal{K}$ induces the multiplicative structure of BGL. It will be proved by showing that the natural map $U\mathcal{K} \to RU\mathcal{K} = U\left(\mathcal{K}^{sf}\right)$ is a stable equivalence of (non-symmetric) motivic spectra. Applying U does not preserve stable equivalences in general.

Consider the shifted motivic symmetric spectrum $\mathcal{K}[1]$ as in [13, p.510]. Its n-th space is $\mathcal{K}_{1+n}$, and $\sigma \in \Sigma_n$ acts on $\mathcal{K}_{1+n}$ by $1 \oplus \sigma \in \Sigma_{1+n}$. The structure map is defined to be the composite

$$\sigma_* : T^p \wedge \mathcal{K}_{1+n} \xrightarrow{\sigma^p} \mathcal{K}_{p+1+n} \xrightarrow{\theta \oplus 1} \mathcal{K}_{1+p+n}$$

where $\theta \in \Sigma_{p+1}$ is the cyclic permutation of order $p+1$. The shifted motivic spectrum $(U\mathcal{K})[1]$ is isomorphic to $U(\mathcal{K}[1])$.

Proposition 5.2.10. *There is a map $\tau : \mathcal{K} \to \mathcal{K}[1]$ of motivic symmetric spectra such that $\tau_n : \mathcal{K}_n \to \mathcal{K}[1]_n$ is a simplicial weak equivalence for every $n \geq 1$.*

Proof. Define $\tau_0 : \mathcal{K}_0 \to \mathcal{K}[1]_0$ by sending the non-basepoint of $\mathcal{K}_0 = S_+$ to $(0, \mathcal{O}) \in \mathcal{K}_1$. Then for each $n \geq 1$ define τ_n to be the composite

$$\mathcal{K}_n \cong K_0 \wedge \mathcal{K}_n \xrightarrow{\tau_0 \wedge 1} \mathcal{K}_1 \wedge \mathcal{K}_n \xrightarrow{\mu} \mathcal{K}_{1+n} = \mathcal{K}[1]_n.$$

It is Σ_n-equivariant since μ is $\Sigma_1 \times \Sigma_n$-equivariant. It is also a simplicial weak equivalence by [7, Lemma 6.3]. To see that these maps define a map of symmetric spectra, it is enough to check that the following diagram commutes.

$$\begin{tikzcd} T \wedge \mathcal{K}_0 \arrow[r, "\sigma"] \arrow[d, "1\wedge\tau"'] & \mathcal{K}_1 \arrow[d, "\tau"] \\ T \wedge \mathcal{K}_1 \arrow[r, "\sigma_*"] & \mathcal{K}_2 \end{tikzcd}$$

The composite map $\sigma\tau$ sends $u \in \mathbb{P}^1$ to the product of $(0, \mathcal{O})$ and $(\mathcal{O}, \mathcal{O}(-1))$. The other composite $\sigma_*(1 \wedge \tau)$ sends u to the product of $(\mathcal{O}, \mathcal{O}(-1))$ and $(0, \mathcal{O})$, then reverses the coordinates. Therefore, the images of u along two composite maps are the same. The same argument works for the unique 1-simplex of T. $\square$

Jardine [13] used the *injective model structure* of motivic symmetric spectra $\mathbf{SM}^\Sigma(S)$ as the preliminary model structure before localizing with respect to stable equivalences. In this model structure, weak equivalences and cofibrations are level equivalences and level cofibrations. Then injective fibrations are defined by the right lifting property with respect to level trivial cofibrations. There is another model structure of $\mathbf{SM}^\Sigma(S)$ called *projective model structure* where weak equivalences and fibrations are defined to be level equivalences and level fibraions. Then projective cofibrations are defined by the left lifting property with respect to level trivial fibrations. The projective model structure for symmetric spectra is explained in [11, 5.1] and the motivic version is explained and used in [19, 2.6.15, 2.6.16]. In particular, there is a level fibrant motivic symmetric spectrum $\mathcal{K}[1]^{\ell f}$ and a level equivalence $j : \mathcal{K}[1] \xrightarrow{\sim} \mathcal{K}[1]^{\ell f}$

Lemma 5.2.11. *The level fibrant model $K[1]^{\ell f}$ of $K[1]$ is stably fibrant.*

Proof. We use the following characterization of stably fibrant symmetric spectra. By definition, a motivic symmetric spectrum A is stably fibrant if the motivic spectrum UA is stably fibrant. Then a motivic spectrum B is stably fibrant if and only if it is level fibrant and the maps $B_n \to \Omega_T B_{n+1}$ induced by structure maps are weak equivalences for all $n \geq 0$ [13, 2.7, 2.8]. Therefore, the lemma follows from Proposition 5.2.10 and Theorem 5.2.7. □

Corollary 5.2.12. *Let $(-)^{sf}$ denote the functorial stably fibrant replacement functor of motivic symmetric spectra. Consider the stably fibrant replacement $i : \mathcal{K} \xrightarrow{\sim} \mathcal{K}^{sf}$ for the K-theory spectrum $\mathcal{K}$. Then i is a level equivalence after the first term. It follows that $Ui : U\mathcal{K} \to U(\mathcal{K}^{sf}) = RU\mathcal{K}$ is a stable equivalence.*

Proof. Consider the commutative diagram of motivic symmetric spectra

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "\tau"] \arrow[d, "i=i_1"'] & \mathcal{K}[1] \arrow[r, "j"] \arrow[d, "i_2"] & \mathcal{K}[1]^{\ell f} \arrow[d, "i_3"] \\
\mathcal{K}^{sf} \arrow[r, "\tilde{\tau}"'] & \mathcal{K}[1]^{sf} \arrow[r, "\tilde{j}"'] & \left(\mathcal{K}[1]^{\ell f}\right)^{sf}
\end{tikzcd}$$

obtained by applying $(-)^{sf}$ to the top row. Three vertical maps are stable equivalences. Since τ and j are level equivalences after the first term, they are stable equivalences. Therefore, $\tilde{\tau}$ and $\tilde{j}$ are stable equivalences as well. Since stable equivalences between stably fibrant motivic symmetric spectra are level equivalences by Corollary 4.6 of [13], we see that $\tilde{\tau}$, $\tilde{j}$, and i_3 are level equivalences. (For i_3, we used Lemma 5.2.11.) Then i_2 is a level equivalence because so is j, and i is a level equivalence after the first term because so is τ. □

Corollary 5.2.13. *Voevodsky's BGL is level equivalent to $RU\mathcal{K}$.*

Proof. In the diagram of Corollary 5.2.12, we showed that i_2 and $\tilde{\tau}$ are level equivalences. Since $BGL = BGL[1]$, by Theorem 5.2.9, there is a motivic spectrum C and level equivalences $U\mathcal{K}[1] \xleftarrow{\sim} C \xrightarrow{\sim} BGL$. Therefore, there is the following zig-zag chain of level equivalences of motivic spectra.

$$RU\mathcal{K} = U\mathcal{K}^{sf} \xrightarrow{U\tilde{\tau}} U\mathcal{K}[1]^{sf} \xleftarrow{Ui_2} U\mathcal{K}[1] \xleftarrow{\sim} C \xrightarrow{\sim} BGL$$

□

The last corollary implies that BGL has a monoidal structure induced by that of $\mathcal{K}$ as discussed earlier. Now we discuss the monoidal structure of BGL defined by Panin, Pimenov, and Röndigs.

Let F_n be the left adjoint to the functor $ev_n : \mathbf{SM}(S) \to \mathbf{M}_\bullet(S)$ sending the motivic spectrum A to its n-th space A_n. Similarly, let F_n^Σ be the left adjoint to the functor $ev_n^\Sigma : \mathbf{SM}^\Sigma(S) \to \mathbf{M}_\bullet(S)$ sending the motivic symmetric spectrum A to its n-th space A_n. By definition, $V\mathrm{F}_n = \mathrm{F}^\Sigma$ since both are left adjoint to ev_n. The functors F_n and F_n^Σ descend, by passing to the homotopy categories, to $\mathrm{F}_n : \mathbf{H}_\bullet(S) \to \mathbf{SH}(S)$ and $\mathrm{F}_n^\Sigma : \mathbf{H}_\bullet(S) \to \mathbf{SH}^\Sigma(S)$. Since any $M \in \mathbf{M}_\bullet(S)$ is cofibrant, $\mathrm{F}_n M$ is a cofibrant spectrum as can be shown by the lifting property. Therefore, $LV\mathrm{F} = \mathrm{F}^\Sigma$. A map $\mu : M \wedge N \to L$ in $\mathbf{M}_\bullet(S)$ induces a map $\mathrm{F}_{n,m}(\mu) : \mathrm{F}_n M \wedge \mathrm{F}_m N \to \mathrm{F}_{n+m} L$ in $\mathbf{SH}(S)$ defined by

$$
\begin{aligned}
RU(LV\mathrm{F}_n M \wedge LV\mathrm{F}_m N) = RU(\mathrm{F}_n^\Sigma M \wedge \mathrm{F}_m^\Sigma N) \cong RU(\mathrm{F}_{n+m}^\Sigma(M \wedge N)) \qquad (5.12)\\
\longrightarrow RU(\mathrm{F}_{n+m}^\Sigma L) = RU(LV\mathrm{F}_{n+m} L) \cong \mathrm{F}_{n+m} L.
\end{aligned}
$$

The isomorphism in the middle follows from the natural isomorphism $\mathrm{F}_n^\Sigma M \wedge \mathrm{F}_m^\Sigma N \cong \mathrm{F}_{n+m}^\Sigma(M \wedge N)$ of Corollary 4.18 of [13]. If there are maps $f : M \to M'$, $g : N \to N'$, and $h : L \to L'$ in $\mathbf{M}_\bullet(S)$ and product maps $\mu : M \wedge N \to L$ and $\nu : M' \wedge N' \to L'$ such that the following diagram commutes

$$
\begin{tikzcd}
M \wedge N \arrow[r, "\mu"] \arrow[d, "f \wedge g"'] & L \arrow[d, "h"] \\
M' \wedge N' \arrow[r, "\nu"] & L'
\end{tikzcd}
$$

in the homotopy category $\mathbf{H}_\bullet(S)$, then the corresponding diagram in $\mathbf{SH}(S)$ shown below commutes as well because F_{n+m}^Σ sends equivalences to level equivalences.

$$
\begin{tikzcd}
\mathrm{F}_n M \wedge \mathrm{F}_m N \arrow[r] \arrow[d] & \mathrm{F}_{n+m} L \arrow[d] \\
\mathrm{F}_n M' \wedge \mathrm{F}_m N' \arrow[r] & \mathrm{F}_{n+m} L'
\end{tikzcd}
$$

Let K^W be the motivic space that assigns to each scheme the loop space of Waldhausen's $S_\bullet$-construction. Then we take a fibrant model $\mathbb{K}^W$ of K^W. Waldhausen multiplication induces a map $\mu_W : \mathbb{K}^W \wedge \mathbb{K}^W \to \mathbb{K}^W$ of pointed motivic spaces. Since there is an isomorphism $\mathbb{K}^W \to \mathbb{Z} \times Gr$ in $\mathbf{H}_\bullet(S)$ [17, 4.3.13], there is a motivic weak equivalence $t : \mathbb{K}^W \to Ex^{\mathbb{A}^1}(\mathbb{Z} \times Gr)$. Let $K^V = Ex^{\mathbb{A}^1}(\mathbb{Z} \times Gr)$. Then there is a product map $\mu_V : K^V \wedge K^V \to K^V$ that coincides with μ_W when we identify K^V with $\mathbb{K}^W$ via i in $\mathbf{H}_\bullet(S)$. Since K^V is the n-th space of BGL for every $n \geq 0$, the identity map on K_V induces a map of spectra $u_n : \mathrm{F}_n K^V \to BGL$ for every $n \geq 0$.

Panin, Pimenov, and Röndigs defined the product map $\mu_{BGL} : BGL \wedge BGL \to BGL$ for $S = \mathrm{Spec}\,(\mathbb{Z})$ to be the unique morphism in the stable homotopy category $\mathbf{SH}(S)$ such that the diagram

$$\begin{tikzcd}
\mathrm{F}_n K^V \wedge \mathrm{F}_n K^V \arrow[r, "\mathrm{F}_{n,n}(\mu_V)"] \arrow[d, "u_n \wedge u_n"'] & \mathrm{F}_{2n} K^V \arrow[d, "u_{2n}"] \\
BGL \wedge BGL \arrow[r, "\mu_{BGL}"] & BGL
\end{tikzcd} \tag{5.13}$$

commutes for every $n \geq 0$ (Theorem 2.2.1 of [20]). The map μ_{BGL} was chosen to be the unique element of $BGL^{0,0}(BGL \wedge BGL)$ that corresponds to $\{u_{2n}\mathrm{F}_{n,n}(\mu_V)\} \in \varprojlim BGL^{4n,2n}((K^V)^{\wedge 2})$ after they prove the isomorphism between $BGL^{0,0}(BGL \wedge BGL)$ and the $\varprojlim$ group. So we may as well say that μ_{BGL} is the unique map making the diagram commute for all sufficiently large n. For any other regular base scheme S, the product map is defined by pulling μ_{BGL} back by the symmetric monoidal functor $\mathbf{SH}(\mathbb{Z}) \to \mathbf{SH}(S)$ induced by $S \to \mathrm{Spec}\,\mathbb{Z}$.

We use their uniqueness assertion to prove the compatibility between the monoidal structure on BGL induced by $\mathcal{K}$ and the one defined by them. By Theorem 5.2.9 and Corollary 5.2.12, there is an isomorphism $\theta : BGL \to RU\mathcal{K}$ in $\mathbf{SH}(S)$, which transfers the multiplication of $\mathcal{K}$ to BGL. If we show that there are morphisms $v_n : \mathrm{F}_n K^V \to RU\mathcal{K}$ for all $n \geq 1$ such that

$$v_n = \theta u_n \tag{5.14}$$

making the diagram

$$\begin{tikzcd}
\mathrm{F}_n K^V \wedge \mathrm{F}_n K^V \arrow[r, "\mathrm{F}_{n,n}(\mu_V)"] \arrow[d, "v_n \wedge v_n"'] & \mathrm{F}_{2n} K^V \arrow[d, "v_{2n}"] \\
RU\mathcal{K} \wedge RU\mathcal{K} \arrow[r, "\mu_{RU\mathcal{K}}"] & RU\mathcal{K}
\end{tikzcd} \tag{5.15}$$

commute, then when we replace μ_{BGL} by the composite

$$BGL \wedge BGL \xrightarrow{\theta \wedge \theta} RU\mathcal{K} \wedge RU\mathcal{K} \xrightarrow{\mu_{RU\mathcal{K}}} RU\mathcal{K} \xrightarrow{\theta^{-1}} BGL$$

in (5.13), the diagram would still commute. Therefore, by their uniqueness assertion, $\theta\mu_{BGL} = \mu_{RU\mathcal{K}}(\theta \wedge \theta)$ showing the compatibility. We will construct v_n and show the commutativity of (5.15) in several steps. The idea is that there is a stable weak equivalence $RU\mathcal{K} \simeq U\mathcal{K} \simeq BGL$, which is a level equivalence after the first term so that the multiplication induced by individual multiplication maps $\mu_{n,n} : \mathcal{K}_n \wedge \mathcal{K}_n \to \mathcal{K}_{2n}$ induce compatibility of the multiplcation of the suspension spectrum generated by

$\mathcal{K}_n$ with the whole spectrum $\mathcal{K}$ and with BGL.

Since the G-construction of K-theory is equivalent to Waldhausen $S_\bullet$-construction, there is an isomorphism $\mathcal{K}_n \to K^W$ in $\mathbf{H}_\bullet(S)$ for every $n \geq 1$. Then it lifts to a weak equivalence $j_n : \mathcal{K}_n \to \mathbb{K}^W$. The isomorphism $w_n : \mathbb{Z} \times Gr \to \mathcal{K}_n$ of Theorem 5.2.8 may be chosen to be the composite in $\mathbf{H}_\bullet(S)$ of the zig-zag chain of equivalences

$$\mathbb{Z} \times Gr \to Ex^{\mathbb{A}^1}(\mathbb{Z} \times Gr) = K^V \xleftarrow{t} \mathbb{K}^W \xleftarrow{j_n} \mathcal{K}_n.$$

Since Waldhausen multiplication μ^W is compatible with the multiplication of G-construction, and μ^V is also derived from μ^W, we have a commutative diagram in $\mathbf{H}_\bullet(S)$ for every $n \geq 1$ where $\mu_{n,n}$ is the multiplication of $\mathcal{K}$ defined in section 5.1.

$$\begin{tikzcd}
\mathcal{K}_n \wedge \mathcal{K}_n \arrow[r, "\mu_{n,n}"] \arrow[d, "j_n t \wedge j_n t"'] & \mathcal{K}_{2n} \arrow[d, "j_{2n} t"] \\
K^V \wedge K^V \arrow[r, "\mu_V"] & K^V
\end{tikzcd}$$

Therefore, we get the following commutative diagram in $\mathbf{SH}(S)$.

$$\begin{tikzcd}
\mathrm{F}_n\mathcal{K}_n \wedge \mathrm{F}_n\mathcal{K}_n \arrow[r, "\mathrm{F}_{n,n}\mu_{n,n}"] \arrow[d, "\mathrm{F}_n j_n t \wedge \mathrm{F}_n j_n t"', "\cong"] & \mathrm{F}_{2n}\mathcal{K}_{2n} \arrow[d, "\mathrm{F}_{2n} j_{2n} t", "\cong"'] \\
\mathrm{F}_n K^V \wedge \mathrm{F}_n K^V \arrow[r, "\mathrm{F}_{n,n}\mu_V"] & \mathrm{F}_{2n} K^V
\end{tikzcd} \tag{5.16}$$

By Theorem 5.2.9, there are stable equivalences $U\mathcal{K} \xleftarrow{\varphi} C \xrightarrow{\psi} BGL$, which are level equivalences after the first term. So we have a commutative diagram for $n \geq 1$ induced by identity maps on n-th spaces.

$$\begin{tikzcd}
\mathrm{F}_n\mathcal{K}_n \arrow[d] & \mathrm{F}_n C_n \arrow[l, "\varphi_n"'] \arrow[r, "\psi_n"] \arrow[d] & \mathrm{F}_n K^V \arrow[d] \\
U\mathcal{K} & C \arrow[l, "\varphi"'] \arrow[r, "\psi"] & BGL
\end{tikzcd}$$

For each $n \geq 1$, the composition $\mathbb{Z} \times Gr \to K^V \xrightarrow{\psi_n^{-1}} C_n \xrightarrow{\varphi_n} \mathcal{K}_n$ in $\mathbf{H}_\bullet(S)$ is w_n of Theorem 5.2.8. Hence, $\psi_n \varphi_n^{-1} = j_n t$, and we get the following commutative diagram in $\mathbf{SH}(S)$ where $i : \mathcal{K} \to \mathcal{K}^{sf}$ is the stably trivial cofibration from $\mathcal{K}$ to its stably fibrant replacement, which is a level equivalence after the first term by Corollary 5.2.12. (We let $\sharp$ denote any map $\mathrm{F}_n X_n \to X$ induced by the identity map on n-th space.)

$$\begin{tikzcd}
\mathrm{F}_n(RU\mathcal{K})_n \arrow[d, "\sharp"'] & \mathrm{F}_n\mathcal{K}_n \arrow[l, "\mathrm{F}_n i_n"', "\cong"] \arrow[r, "\mathrm{F}_n j_n t", "\cong"'] \arrow[d, "\sharp"'] & \mathrm{F}_n K^V \arrow[d, "u_n"] \\
RU\mathcal{K} & U\mathcal{K} \arrow[l, "Ui"', "\cong"] \arrow[r, "\psi\varphi^{-1}", "\cong"'] & BGL
\end{tikzcd}$$

The composition of bottom maps is the isomorphism $\theta : BGL \to RU\mathcal{K}$ transfering the multiplication. Define $v_n : \mathrm{F}_n K^V \to RU\mathcal{K}$ to be the composite of the arrows of the top row with the leftmost vertical map.

$$v_n : \mathrm{F}_n K^V \xrightarrow{(\mathrm{F}_n j_n t)^{-1}} \mathrm{F}_n\mathcal{K}_n \xrightarrow{\mathrm{F}_n i_n} \mathrm{F}_n(RU\mathcal{K})_n \xrightarrow{\sharp} RU\mathcal{K} \tag{5.17}$$

Then $v_n = \theta u_n$ as in (5.14). It remains to show the commutativity of the diagram (5.15).

There is a map $k_n : \mathrm{F}^{\Sigma}_n\mathcal{K}_n \to \mathcal{K}$ induced by the identity map on $\mathcal{K}_n$, and the diagram

$$\begin{tikzcd}
\mathrm{F}^{\Sigma}_n\mathcal{K}_n \wedge \mathrm{F}^{\Sigma}_n\mathcal{K}_n \arrow[r, "\cong"] \arrow[d, "k_n \wedge k_n"'] & \mathrm{F}^{\Sigma}_{2n}(\mathcal{K}_n \wedge \mathcal{K}_n) \arrow[r, "\mathrm{F}^{\Sigma}_{2n}\mu_{n,n}"] & \mathrm{F}^{\Sigma}_{2n}\mathcal{K}_{2n} \arrow[d, "k_{2n}"] \\
\mathcal{K} \wedge \mathcal{K} \arrow[rr, "\mu"] & & \mathcal{K}
\end{tikzcd}$$

commutes since $k_{2n}\mathrm{F}^{\Sigma}_{2n}\mu_{n,n}$ is induced by the map $\mu_{n,n} : \mathcal{K}_n \wedge \mathcal{K}_n \to \mathcal{K}_{2n}$, which is a component of the map $\mu : \mathcal{K} \wedge \mathcal{K} \to \mathcal{K}$. We pass this diagram to the homotopy category $\mathbf{SH}^{\Sigma}(S)$, then apply RU. Then we get the following commutative diagram. (See (5.12).)

$$\begin{tikzcd}
\mathrm{F}_n\mathcal{K}_n \wedge \mathrm{F}_n\mathcal{K}_n \arrow[r, "\mathrm{F}_{n,n}\mu_{n,n}"] \arrow[d, "RU(k_n \wedge k_n)"'] & \mathrm{F}_{2n}\mathcal{K}_{2n} \arrow[r, "\cong"] & RU(\mathrm{F}^{\Sigma}_{2n}\mathcal{K}_{2n}) \arrow[d, "RUk_{2n}"] \\
RU(\mathcal{K} \wedge \mathcal{K}) \arrow[rr, "RU\mu"] & & RU\mathcal{K}
\end{tikzcd} \tag{5.18}$$

Recall the way the derived adjunction

$$\xi : \mathrm{Hom}_{\mathbf{SH}^{\Sigma}(S)}(LV-, -) \rightleftarrows \mathrm{Hom}_{\mathbf{SH}(S)}(-, RU-) : \chi$$

is defined. Denoting $(-)^{co}$ for cofibrant replacement and $(-)^{sf}$ for stably fibrant replacement, the left Hom set is identified with $\mathrm{Hom}_{\mathbf{SM}^{\Sigma}(S)}\left(V(-)^{co}, (-)^{sf}\right)/\sim$ and the right Hom set is identified with $\mathrm{Hom}_{\mathbf{SM}(S)}\left((-)^{co}, U(-)^{sf}\right)/\sim$. Then the adjunction (V, U) defines ξ and χ.

$$\mathrm{Hom}_{\mathbf{SM}^{\Sigma}(S)}\left(V(-)^{co}, (-)^{sf}\right)/\sim \;\rightleftarrows\; \mathrm{Hom}_{\mathbf{SM}(S)}\left((-)^{co}, U(-)^{sf}\right)/\sim$$

Applying this to the map $\sharp : \mathrm{F}_n(RU\mathcal{K})_n \to RU\mathcal{K}$ induced by the identity map on $(RU\mathcal{K})_n$, we see that $\chi\sharp$ is the composite

$$\chi\sharp : \mathrm{F}^{\Sigma}_n(RU\mathcal{K})_n \xrightarrow{l_n} (RU\mathcal{K}) \xrightarrow{i^{-1}} \mathcal{K}$$

where l_n is the map induced by the identity on $(RU\mathcal{K})_n$. Since the left diagram below commutes, by taking adjoint we get the commutativity of the right diagram.

$$\begin{tikzcd}
\mathrm{F}^{\Sigma}_n\mathcal{K}_n \arrow[r, equal] \arrow[d, "\mathrm{F}^{\Sigma}_n i_n"'] & \mathrm{F}^{\Sigma}_n\mathcal{K}_n \arrow[d, "k_n"] \\
\mathrm{F}^{\Sigma}_n(RU\mathcal{K})_n \arrow[r, "\chi\sharp"] & \mathcal{K}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{F}_n\mathcal{K}_n \arrow[r, "\cong"] \arrow[d, "\mathrm{F}_n i_n"'] & RU(\mathrm{F}^{\Sigma}_n\mathcal{K}_n) \arrow[d, "RUk_n"] \\
\mathrm{F}_n(RU\mathcal{K})_n \arrow[r, "\sharp"] & RU\mathcal{K}
\end{tikzcd}$$

Therefore, the upper right corner of the diagram (5.18) can be replaced by $F_{2n}(RU\mathcal{K})_{2n}$.

$$\begin{tikzcd}
\mathrm{F}_n\mathcal{K}_n \wedge \mathrm{F}_n\mathcal{K}_n \arrow[r, "\mathrm{F}_{n,n}\mu_{n,n}"] \arrow[d, "RU(k_n\wedge k_n)"'] & \mathrm{F}_{2n}\mathcal{K}_{2n} \arrow[r, "\mathrm{F}_{2n}i_{2n}"] & F_{2n}(RU\mathcal{K})_{2n} \arrow[d, "\sharp"] \\
RU(\mathcal{K}\wedge\mathcal{K}) \arrow[rr, "RU\mu"] & & RU\mathcal{K}
\end{tikzcd} \tag{5.19}$$

Similarly, applying the adjointness argument to k_n, we see that $\xi k_n : \mathrm{F}_n\mathcal{K}_n \to RU\mathcal{K}_n$ is the map induced by $i_n : \mathcal{K}_n \to (\mathcal{K}^{sf})_n$. Thus we have the following commutative diagram where the left map is induced by the identity on $(RU\mathcal{K})_n$.

$$\begin{tikzcd}
RU\mathcal{K} \arrow[d, equal, "1"'] & \mathrm{F}_n(RU\mathcal{K})_n \arrow[l, "\sharp"'] & \mathrm{F}_n\mathcal{K}_n \arrow[l, "\mathrm{F}_n i_n"'] \arrow[d, "\xi k_n"] \\
RU\mathcal{K} \arrow[rr, equal, "RU1"] & & RU\mathcal{K}
\end{tikzcd}$$

Taking the adjoint diagram, we get

$$\begin{tikzcd}
\mathrm{F}^{\Sigma}_n(RU\mathcal{K})_n \arrow[d, "LV\sharp"'] & \mathrm{F}^{\Sigma}_n\mathcal{K}_n \arrow[l, "\mathrm{F}^{\Sigma}_n i_n"'] \arrow[d, "k_n"] \\
LVRU\mathcal{K} \arrow[r, "\cong"] & \mathcal{K}
\end{tikzcd}$$

where the bottom isomorphism is the natural isomorphism $LVRU \cong 1$. Smashing each object of the diagram with itself, we get

$$\begin{tikzcd}
\mathrm{F}^{\Sigma}_n(RU\mathcal{K})_n \wedge \mathrm{F}^{\Sigma}_n(RU\mathcal{K})_n \arrow[d, "LV\sharp\wedge LV\sharp"'] & \mathrm{F}^{\Sigma}_n\mathcal{K}_n \wedge \mathrm{F}^{\Sigma}_n\mathcal{K}_n \arrow[l, "\mathrm{F}^{\Sigma}_n i_n\wedge \mathrm{F}^{\Sigma}_n i_n"'] \arrow[d, "k_n\wedge k_n"] \\
LVRU\mathcal{K} \wedge LVRU\mathcal{K} \arrow[r, "\cong"] & \mathcal{K}\wedge\mathcal{K}
\end{tikzcd}$$

Finally, apply RU.

$$\begin{tikzcd}
\mathrm{F}_n(RU\mathcal{K})_n \wedge \mathrm{F}_n(RU\mathcal{K})_n \arrow[d, "\sharp\wedge\sharp"'] & \mathrm{F}_n\mathcal{K}_n \wedge \mathrm{F}_n\mathcal{K}_n \arrow[l, "\mathrm{F}_n i_n \wedge \mathrm{F}_n i_n"'] \arrow[d, "RU(k_n\wedge k_n)"] \\
RU\mathcal{K} \wedge RU\mathcal{K} \arrow[r, "\cong"] & RU(\mathcal{K}\wedge\mathcal{K})
\end{tikzcd} \tag{5.20}$$

Attach diagrams (5.20) and (5.16) to the diagram (5.19). Then we get

$$\begin{tikzcd}
 & \mathrm{F}_n K^V \wedge \mathrm{F}_n K^V \arrow[r, "\mathrm{F}_{n,n}\mu^V"] & \mathrm{F}_{2n} K^V & \\
\mathrm{F}_n(RU\mathcal{K}) \wedge \mathrm{F}_n(RU\mathcal{K}) \arrow[d, "\sharp\wedge\sharp"'] & \mathrm{F}_n\mathcal{K}_n \wedge \mathrm{F}_n\mathcal{K}_n \arrow[l, "\mathrm{F}_n i_n \wedge \mathrm{F}_n i_n"] \arrow[u, "\mathrm{F}_n j_n t\wedge \mathrm{F}_n j_n t", "\cong"'] \arrow[r, "\mathrm{F}_{n,n}\mu_{n,n}"] \arrow[d, "RU(k_n\wedge k_n)"] & \mathrm{F}_{2n}\mathcal{K}_{2n} \arrow[u, "\cong", "\mathrm{F}_{2n} j_{2n} t"'] \arrow[r, "\mathrm{F}_{2n} i_{2n}"'] & \mathrm{F}_{2n}(RU\mathcal{K})_{2n} \arrow[d, "\sharp"] \\
RU\mathcal{K} \wedge RU\mathcal{K} \arrow[r, "\cong"'] & RU(\mathcal{K}\wedge\mathcal{K}) \arrow[rr, "RU\mu"'] & & RU\mathcal{K}
\end{tikzcd}$$

By the definition of $\mu_{RU\mathcal{K}}$ at (5.11) and the definition of v_n at (5.17), this diagram proves the commutativity of (5.15).

Theorem 5.2.14. *If the base scheme S is regular, then the motivic symmetric ring spectrum $\mathcal{K}$ constructed in section 5.1 induces multiplication of BGL in the stable homotopy category $\mathbf{SH}(S)$, and this multiplication is compatible with the one defined by Panin, Pimenov, and Röndigs in [20]. Therefore, $\mathcal{K}$ and BGL are isomorphic as homotopy ring spectra.*

Proof. We have proved that $\mathcal{K}$ induces multiplication of BGL for any regular S. We also have proved the compatibility for $S = \mathrm{Spec}\,(\mathbb{Z})$. For any other regular scheme S, the multiplication by Panin, et al. is the pullback of the multiplication for $\mathrm{Spec}\,(\mathbb{Z})$ along the functor $f^* : \mathbf{SH}(\mathbb{Z}) \to \mathbf{SH}(S)$ induced by $f : S \to \mathrm{Spec}\,(\mathbb{Z})$. In that case, their multiplication is compatible with the multiplication of $f^*\mathcal{K}_{\mathbb{Z}}$. But the multiplication of $f^*\mathcal{K}_{\mathbb{Z}}$ is compatible with $\mathcal{K}_S$ by Theorem 5.1.9. □

There is Voevodsky's theorem that says BGL represents Weibel's homotopy K-theory for any Noetherian base scheme [24, 6.9], whose proof can be found in [3, 2.15]. Showing that our spectrum also represents Weibel's homotopy K-theory could be a starting point in showing the equivalence of our spectrum $\mathcal{K}$ and Voevodsky's BGL for non-regular noetherian base scheme S.

References

[1] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos.* Lecture Notes in Mathematics, Vol. 269. Springer-Verlag, Berlin, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[2] M. Artin, J. E. Bertin, M. Demazure, P. Gabriel, A. Grothendieck, M. Raynaud, and J.-P. Serre. *Schémas en groupes. Fasc. 1: Exposés 1 à 4*, volume 1963 of *Séminaire de Géométrie Algébrique de l'Institut des Hautes Études Scientifiques.* Institut des Hautes Études Scientifiques, Paris, 1963/1964.

[3] Denis-Charles Cisinski. Descente propre en K-théorie invariante par homotopie. `arXiv:1003.1487 [math.AG]`.

[4] W. G. Dwyer and J. Spaliński. Homotopy theories and model categories. In *Handbook of algebraic topology*, pages 73–126. North-Holland, Amsterdam, 1995.

[5] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli. *Fundamental algebraic geometry*, volume 123 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2005. Grothendieck's FGA explained.

[6] Eric M. Friedlander and Andrei Suslin. The spectral sequence relating algebraic K-theory to motivic cohomology. *Ann. Sci. École Norm. Sup. (4)*, 35(6):773–875, 2002.

[7] Henri Gillet and Daniel R. Grayson. The loop space of the Q-construction. *Illinois J. Math.*, 31(4):574–597, 1987.

[8] Daniel R. Grayson. Algebraic K-theory. Lecture Notes, 2003, http://www.math.uiuc.edu/ dan/Courses/2003/Spring/416/.

[9] Daniel R. Grayson. Weight filtrations via commuting automorphisms. *K-Theory*, 9(2):139–172, 1995.

[10] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1999.

[11] Mark Hovey, Brooke Shipley, and Jeff Smith. Symmetric spectra. *J. Amer. Math. Soc.*, 13(1):149–208, 2000.

[12] J. F. Jardine. Simplicial presheaves. *J. Pure Appl. Algebra*, 47(1):35–87, 1987.

[13] J. F. Jardine. Motivic symmetric spectra. *Doc. Math.*, 5:445–553 (electronic), 2000.

[14] Gregory Maxwell Kelly. *Basic concepts of enriched category theory*, volume 64 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1982.

[15] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1998.

[16] Stephen A. Mitchell. Hypercohomology spectra and Thomason's descent theorem. In *Algebraic K-theory (Toronto, ON, 1996)*, volume 16 of *Fields Inst. Commun.*, pages 221–277. Amer. Math. Soc., Providence, RI, 1997.

[17] Fabien Morel and Vladimir Voevodsky. $\mathbf{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, (90):45–143 (2001), 1999.

[18] Oliver Roendigs, Markus Spitzweck, and Paul Arne Ostvaer. Motivic strict ring models for K-theory. Proc. Amer. Math. Soc., posted on May 10, 2010, PII S 0002-9939(10)10394-3 (to appear in print).

[19] Pablo Pelaez. On the functoriality of the slice filtration. Preprint, February 1, 2010, K-theory Preprint Archives, http://www.math.uiuc.edu/K-theory/0950/.

[20] Ivan Panin, Konstantin Pimenov, and Oliver Röndigs. On *Voevodsky*'s algebraic K-theory spectrum. In *Algebraic topology*, volume 4 of *Abel Symp.*, pages 279–330. Springer, Berlin, 2009.

[21] Daniel Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

[22] Stefan Schwede. An untitled book project about symmetric spectra. Preprint, July 12, 2007, http://www.math.uni-bonn.de/people/schwede/.

[23] R. W. Thomason and Thomas Trobaugh. Higher algebraic K-theory of schemes and of derived categories. In *The Grothendieck Festschrift, Vol. III*, volume 88 of *Progr. Math.*, pages 247–435. Birkhäuser Boston, Boston, MA, 1990.

[24] Vladimir Voevodsky. $\mathbf{A}^1$-homotopy theory. In *Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998)*, number Extra Vol. I, pages 579–604 (electronic), 1998.